Pauta Aux 5

P1) $T = (T_1, \ldots, T_n)$; $T_i \sim e^{-(t_i - \theta)} \mathbb{1}_{t_i \geq \theta}$; $\pi(\theta) = e^{-\theta} \mathbb{1}_{[0,\infty)}$

calcular el estimador MAP

NOTA: $\prod_{i=1}^{n} \mathbb{1}_{A_i} = \mathbb{1}_{\bigcap_{i=1}^{n} A_i}$

$$f(t \mid \theta) = e^{-\sum_{i=1}^{n}(t_i - \theta)} \prod_{i=1}^{n} \mathbb{1}_{t_i \geq \theta}$$

$$= e^{-\sum_{i=1}^{n}(t_i - \theta)} \mathbb{1}_{\min\{t_i\} \geq \theta}$$

$$\Rightarrow f(\theta \mid t) = \frac{e^{-\sum_{i=1}^{n}(t_i - \theta)} \mathbb{1}_{\min\{t_i\} \geq \theta} \cdot e^{-\theta} \mathbb{1}_{[0,\infty)}}{f(t)} \longleftarrow \text{¡NO depende de } \theta\text{!}$$

$$= \frac{e^{-\sum t_i} \cdot e^{(n-1)\theta} \mathbb{1}_{\min\{t_i\} \geq \theta \geq 0}}{f(t)}$$

$$\Rightarrow \hat{\theta}_{MAP} \in \underset{\theta \in \Theta}{\operatorname{argmax}} f(\theta \mid t) \Rightarrow \hat{\theta}_{MAP} = \min_{1 \leq i \leq n}\{t_i\}$$

P2) $X = (X_1, \ldots, X_n)$ MAS ; $X_i \sim \exp(\theta)$

a) $T(X) = \frac{2}{\theta}\sum_{i=1}^{n} X_i$ es pivotal y $T(X) \sim \chi^2(2n)$

·) Es decreciente c/r a θ ⇒ Monótono

·) $\sum_{i=1}^{n} X_i \sim \Gamma(n, \theta) \Rightarrow f_{\sum X_i}(t) = \frac{1}{\Gamma(n)\,\theta^n} t^{n-1} e^{-t/\theta} \mathbb{1}_{t \geq 0}$

Luego $f_{\frac{2}{\theta}\sum X_i}(t) = \frac{\theta}{2} f_{\sum X_i}\left(\frac{\theta t}{2}\right)$

$$= \frac{\theta}{2} \frac{1}{\Gamma(n)\,\theta^n} \left(\frac{\theta t}{2}\right)^{n-1} e^{-\frac{\theta t}{2\theta}} \mathbb{1}_{t\geq 0}$$

$$= \frac{1}{\Gamma(n)\,2^n} t^{n-1} e^{-t/2} \mathbb{1}_{t\geq 0}$$ que es justamente una chi-cuadrado a $2n$ grados de libertad

NO depende de θ

$\Rightarrow T(X)$ es pivote

b) Encontrar intervalo de confianza de 95% para θ

Sean t_1, t_2 tales que

$$\mathbb{P}(t_1 \leq T(X) \leq t_2) = 1-\alpha = 0{,}95 \quad (\text{luego } \alpha = 0{,}05)$$

$$\mathbb{P}(T \leq t_2) - \mathbb{P}(T \leq t_1) = \int_{t_1}^{t_2} f_T(x)\,dx = 1-\alpha$$

Se puede tomar, por ejemplo, $t_1 = F_T^{-1}(\alpha)$ (inversa de la distribución $\chi^2(2n)$)

$t_2 = \infty$

Ahora $t_1 \leq T(X) \leq t_2 \Leftrightarrow \boxed{\frac{2\sum_{i=1}^n X_i}{t_2} \leq \theta \leq \frac{2\sum_{i=1}^n X_i}{t_1}}$

O sea, $\boxed{0 \leq \theta \leq \frac{2\sum_{i=1}^n X_i}{F_T^{-1}(\alpha)}}$

P 3) $X = (X_1, \dots, X_n)$; $X_i \sim \mathbb{P}_\theta$ con densidad $f(x,\theta)$

$$r(x,\theta,h) := \frac{f(x,\theta+h) - f(x,\theta)}{f(x,\theta)}$$

$I_{\theta,h} = \mathbb{E}_\theta((r(X,\theta,h))^2)$, $\hat\theta$ insesgado.

a) Demostrar que $Var(\hat\theta) \geq \dfrac{h^2}{I_{\theta,h}}$

Primero, $\mathbb{E}(r(X,\theta,h)) = \displaystyle\int_{\mathbb{R}} \frac{f(x,\theta+h) - f(x,\theta)}{\cancel{f(x,\theta)}} \cdot \cancel{f(x,\theta)}\, dx$

$$= \int_{\mathbb{R}} f(x,\theta+h)\,dx - \int_{\mathbb{R}} f(x,\theta)\,d\theta = 1 - 1 = 0$$

$\Rightarrow Var(r(X,\theta,h)) = I_{\theta,h}$

Ahora $Cov(r(x,\theta,h),\hat\theta) = \mathbb{E}[(r - \mathbb{E}(r))(\hat\theta - \mathbb{E}\hat\theta)]$

$$= \mathbb{E}(r\hat\theta)$$

$$= \int \hat\theta f(x,\theta+h)\,dx - \int \hat\theta f(x,\theta)\,dx = \theta + h - \theta = h$$

Por Cauchy-Schwarz

$h^2 = Cov(r,\hat\theta)^2 \leq \cancel{Var}^{\,I_{\theta,h}}(r) \cdot Var(\hat\theta)$

$$\Rightarrow Var(\hat\theta) \geq \frac{h^2}{I_{\theta,h}}$$

b) Calcular $I_{\theta,h}$ para $X_i \sim Unif[0,\theta]$; $\theta > 0$

$$I_{\theta,h} = \mathbb{E}(r^2) \quad \text{(Notar que } f(x,\theta) = \frac{1}{\theta^n} \mathbb{1}_{[0,\theta]}\text{)}$$

$$= \int \left[\frac{\frac{1}{(\theta+h)^n} - \frac{1}{\theta^n}}{\frac{1}{\theta^n}} \right]^2 \frac{1}{\theta^n} \mathbb{1}_{[0,\theta]}(x_1, \dots, x_n)\, dx_1 \dots dx_n$$

$$= \left[\frac{\frac{1}{(\theta+h)^n} - \frac{1}{\theta^n}}{\frac{1}{\theta^n}} \right]^2 = \left[\frac{\theta^n - (\theta+h)^n}{(\theta+h)^n} \right]^2$$