Resumen Calculo Vectorial.

* Vamos a diferenciar dos tipos de funciones.

1) Funciones escalares (escalar) $f : \mathbb{R}^3 \to \mathbb{R}$.

Las vamos a anotar en minuscula.

2) Funciones vectoriales (Vectores) $F : \mathbb{R}^3 \to \mathbb{R}^3$

Las vamos a anotar en mayuscula.

* Sistema de Coordenadas

Diremos que la función vectorial $\vec{r} : \mathbb{R}^3 \to \mathbb{R}^3$ es una Transformación si cumple que.

$$\vec{r}(u, v, w) = \left(x(u,v,w), y(u,v,w), z(u,v,w) \right)$$

y el Jacobiano sea invertible $\forall$ u, v, w

$$J\vec{r}(u_0, v_0, w_0) = \left[\frac{\partial \vec{r}}{\partial u} \quad \frac{\partial \vec{r}}{\partial v} \quad \frac{\partial \vec{r}}{\partial w} \right]$$

PROARTE

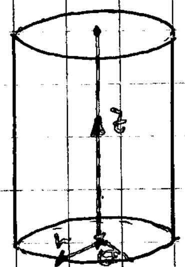
Vector unitario $\hat{u} = \dfrac{\partial \vec{r}}{\partial u} \Big/ \left\| \dfrac{\partial \vec{r}}{\partial u} \right\|$

Luego, determinaremos $h_u = \left\| \dfrac{\partial \vec{r}}{\partial u} \right\|$:

$\dfrac{\partial \vec{r}}{\partial u} = h_u \hat{u}$, esto es igual para v, y w.

Ex Coordenadas cilíndricas.

$$\vec{r}(r, \theta, z) = (r\cos\theta,\ r\operatorname{sen}\theta,\ z)$$



$z \in (-\infty, +\infty)$
$r \in (0, +\infty)$
$\theta \in [0, 2\pi)$

$\dfrac{\partial \vec{r}}{\partial r} = (\cos\theta, \operatorname{sen}\theta)$, $h_r = 1$

$\dfrac{\partial \vec{r}}{\partial \theta} = (-r\operatorname{sen}\theta, r\cos\theta)$, $h_\theta = r$

$\dfrac{\partial \vec{r}}{\partial z} = (0, 0, 1)$, $h_z = 1$

$$\hat{r} = (\cos\theta, \sin\theta, 0)$$

$$\hat{\theta} = (-\sin\theta, \cos\theta, 0)$$

$$\hat{z} = (0, 0, 1)$$

Ex Coordenadas esfericas

$$\vec{r} = (r \sin\varphi \cos\theta, r \sin\varphi \sin\theta, r\cos\varphi) = \vec{r}(r, \theta, \varphi).$$

$$\frac{\partial \vec{r}}{\partial r} = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi), \quad h_r = 1$$

$$\frac{\partial \vec{r}}{\partial \theta} = (-r\sin\varphi\sin\theta, r\sin\varphi\cos\theta, 0), \quad h_\theta = r\sin\varphi$$

$$\frac{\partial \vec{r}}{\partial \varphi} = (r\cos\varphi\cos\theta, r\cos\varphi\sin\theta, -r\sin\varphi), \quad h_\varphi = r$$

Diferenciales.

Gradiente

$$gr(f) = \nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}\hat{u} + \frac{1}{h_v}\frac{\partial f}{\partial v}\hat{v} + \frac{1}{h_w}\frac{\partial f}{\partial w}\hat{w}$$

divergencia

$$div(\overline{F}) = \nabla \circ \overline{F} = \frac{1}{h_u h_v h_w}\left\{\frac{\partial}{\partial u}(\overline{F}_u h_v h_w) + \frac{\partial}{\partial v}(\overline{F}_v h_u h_w) + \frac{\partial}{\partial w}(\overline{F}_w h_u h_v)\right\}$$

Rotor

$$Rot(\overline{F}) = \nabla \times \overline{F} = \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u \hat{u} & h_v \hat{v} & h_w \hat{w} \\ \partial/\partial u & \partial/\partial v & \partial/\partial w \\ \overline{F}_u h_u & \overline{F}_v h_v & \overline{F}_w h_w \end{vmatrix}$$

④ Jacobiano e integración con cambio de variable.

$$\int_{f(D)} g(x)\,dx = \int_{\Omega} g(f(y))\,\| J(f(y))\|\,dy$$

donde Ω es un volumen de integración

f, es una transformación.

$$J(f(y)) = \left[\frac{\partial f}{\partial y_1} \middle| \frac{\partial f}{\partial y_2} \middle| \cdots \right]$$

④ Cómo definir un volumen, una superficie y una línea.

Superficie.

Una superficie es $\vec{\varphi} : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$

$S = \{ \vec{\varphi}(u,v) : u, v \in D \}$

$$\hat{t}_u = \frac{\partial \vec{\varphi}}{\partial u} \Big/ \left\| \frac{\partial \vec{\varphi}}{\partial u} \right\| , \quad \hat{t}_v = \frac{\partial \vec{\varphi}}{\partial v} \Big/ \left\| \frac{\partial \vec{\varphi}}{\partial v} \right\|$$

$$\hat{n} = \frac{\hat{t}_u \times \hat{t}_v}{\| \hat{t}_u \times \hat{t}_v \|}$$

↳ Línea.

Una línea se define como, $\vec{r} : [a, b] \rightarrow \mathbb{R}^3$.

Sea $\vec{F}$, vector, se define la integral de trabajo

$$\int_{\Gamma} \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}(t)\, dt.$$

* Flujo

Se define el flujo del vector $\vec{F}$ como.

$$\Phi = \iint_S \vec{F} \cdot \hat{n} \cdot d\vec{S}$$

Nota → cuando superficie es cerrada, se esta delimitando un volumen.

Teo de la divergencia.

$$\iint_{\partial\Omega} \vec{F} \cdot d\vec{S} = \iiint_{\Omega} div(\vec{F})\, dV$$

— o —

Electricidad en el Vacío

$$\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{Q_1 Q_2}{\|\vec{r}\|^3} \vec{r}$$

la Fuerza que siente Q_2 debido a la presencia de Q_1

ε_0 → permitividad del vacío

$$\varepsilon_0 = \frac{10^7}{4\pi c^2} = 8{,}8541 \times 10^{-12} \left(\frac{F}{m}\right)$$

Auxilios

P.)

b) $\displaystyle\iint_C \log(x^2+y^2)\,dx\,dy$.

$C = \{ (x,y) : a^2 \leq x^2+y^2 \leq b^2 ,\ x, y \geq 0 \}$.

En efecto primero hacemos la Transformación (S.C.U)

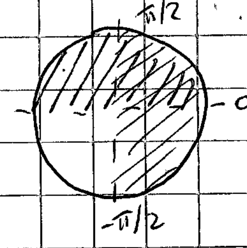
$\vec{r}(r,\theta) = (r\cos\theta, r\,\text{sen}\,\theta)$

$x = r\cos\theta$
$y = r\,\text{sen}\,\theta$

$\vec{r}^{-1}(C) = \{ (r,\theta) : a^2 \leq r^2 \leq b^2 ,\ r \geq 0,\ \cos\theta \geq 0,\ \text{sen}\,\theta \geq 0 \}$.

$r \in (a,b)$

$\theta \in (0, \pi/2)$



/// $\cos\theta \geq 0$
// $\text{sen}\,\theta \geq 0$

$$\int_a^b \int_0^{\pi/2} \log(r^2)\ \| J_{\vec{r}}(r,\theta) \|\ dr\,d\theta$$

$$J\vec{r}(r,\theta) = \begin{bmatrix} \partial x/\partial r & x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{bmatrix}$$

$$J\vec{r}(r,\theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$\| J\vec{r}(r,\theta) \| = r\cos^2\theta - (-r\sin^2\theta) = r$$

$$\int_a^b \int_0^{\pi/2} \log(r^2)\, r\, dr\, d\theta. = \pi/2 \int_a^b \log(r^2)\, r\, dr$$

C.V , $u = r^2$

$du = 2r\,dr$

$$= \frac{\pi}{2}\frac{1}{2}\int_{a^2}^{b^2} \log(u)\, du = \frac{\pi}{4} \left\{ u\left(\log(u) - 1\right) \Big|_{a^2}^{b^2} \right\}$$

P2) En efecto, $\vec{F}(x,y,z) = (4xz, xyz, 3z)$

$\Omega = \{ (x,y,z) \in \mathbb{R}^3 : \sqrt{x^2+y^2} \leq z,\ y \geq 0,\ 0 \leq z \leq 4 \}$.

Calculamos $\displaystyle\iint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S}_1 + \iint_{S_2} \vec{F} \cdot d\vec{S}_2$

$\displaystyle + \iint_{S_3} \vec{F} \cdot d\vec{S}_3$

S_1: $\vec{r}_1 : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$

$\vec{r}_1 = (r\cos\theta, r\sin\theta, r)$

$\vec{F} = (4r^2\cos\theta, r^3\cos\theta\sin\theta, 3r)$

límite, $r = z \Rightarrow r \in (0,4)$

$y \geq 0 \Rightarrow r\sin\theta \geq 0 \Rightarrow \theta \in (0,\pi)$

$d\vec{S}_1 = \hat{n}\ h_u h_v\ du\, dv$

$u = r \ , \quad h_r = 1 \ ,$
$v = \theta \quad\quad h_\theta = r$

$d\vec{S}_1 = \hat{n}\ r\, dr\, d\theta .$

$\hat{n} = \dfrac{\hat{u} \times \hat{v}}{\| \hat{u} \times \hat{v} \|}$

$\hat{u} = \dfrac{\partial \vec{r}_1}{\partial r} = (\cos\theta,\ \sin\theta,\ 1)$

$\hat{v} = \dfrac{\partial \vec{r}_2}{\partial \theta} = (-\sin\theta,\ \cos\theta,\ 0)$

$\hat{u} \times \hat{v} = (-\cos\theta\ ,\ -\sin\theta\ ,\ 1)$

$\hat{n} = (\cos\theta,\ \sin\theta,\ -1) = \hat{r} - \hat{k}$

$$\int_0^4\int_0^\pi \left(4r^2\cos\theta,\ r^3\cos\theta\sin\theta,\ 3r\right)\left(\cos\theta,\ \sin\theta,\ -1\right)\ r\,dr\,d\theta$$

$$\int_0^4\int_0^\pi \underset{①}{4r^3\cos^2\theta} + \underset{②}{r^4\cos\theta\sin^2\theta} \underset{③}{- 3r^2}\ dr\,d\theta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}, \qquad \cos^2\theta + \sin^2\theta = 1$$

$$\sin 2\theta = 2\cos\theta\sin\theta.$$

① $$\int_0^4 \frac{4r^3}{2}\left\{\int_0^\pi d\theta + \int_0^\pi \cos 2\theta\right\}$$

$$\int_0^4 \frac{4r^3}{2}\left(\pi + \overset{0}{\frac{\sin 2\theta}{2}\Bigg|_0^\pi}\right)$$

$$\int_0^4 2r^3\pi = \frac{r^4}{2}\pi\Bigg|_0^4 = 2\pi\frac{4^4}{4} = \boxed{2\pi\, 4^3}$$

② $$\int_0^4 r^4 \overset{0}{\int_0^\pi \cos\theta\sin^2\theta} = 0$$

$$\int u\,dv = uv - \int du\,v$$

$$\int_0^{\pi} \underbrace{\cos\theta}_{dv}\ \underbrace{\sin^2\theta}_{u}\ d\theta$$

$du = 2\sin\theta\cos\theta.$

$v = \sin\theta.$

$$= \sin^3\theta\Big|_0^{\pi} \xrightarrow{} 0 \quad - 2\int_0^{\pi} \sin^2\theta\cos\theta$$

$$\int_0^{\pi} \cos\theta\sin^2\theta\, d\theta = -2\int_0^{\pi} \sin^2\cos\theta \Rightarrow 0.$$

(3) $= -\int_0^{4}\int_0^{\pi} 3r^2\, dr\, d\theta$

$$= \cancel{3}\pi\frac{r^3}{\cancel{3}} = \pi r^3\Big|_0^4 = \pi 4^3$$

$$\iint_{S_1} \vec{F}\cdot d\vec{S}_1 = 2\pi 4^3 - \pi 4^3 = \boxed{64\pi}$$

S_2: Topo.

$z = 4$
$\theta \in (0, \pi)$
$r \in (0, 4)$

$\vec{r_2} = (r\cos\theta, r\sin\theta, 4)$

$\vec{r_2} = r\hat{r} + 4\hat{k}$

$\hat{n} = \hat{k}$

$\vec{F} = (4r\cos\theta\, 4, 4r^2\cos\theta\sin\theta, 12)$

$d\vec{s} = r\,dr\,d\theta\,\hat{k}$

$$\int_0^4\int_0^\pi 12r\,dr\,d\theta = 12\pi\int_0^4 r\,dr = 12\pi\frac{r^2}{2}$$

$$6\pi 4^2 = \boxed{6 \cdot 4^2\pi}$$

$S_3 : \hat{r}_3 = (x, 0, z)$

$\vec{F} = (4xz, 0, 3z)$

$\hat{n} = \hat{y}$.

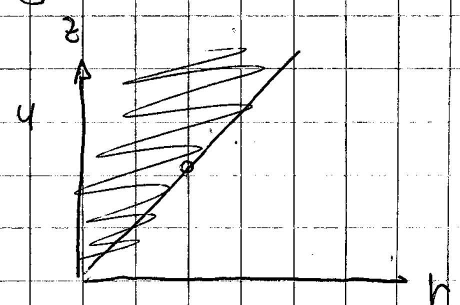
$$\iint_{S_3} \vec{F} \cdot d\vec{s} = 0$$

luego $\boxed{\iint_S \vec{F} \cdot d\vec{s} = 160\pi}$

ahora lo hacemos con la divergencia.

$\vec{r} : (r\cos\theta, r\sin\theta, z)$

$\theta \in (0, \pi)$
$r \in (0, 4)$
$z \in (r, 4)$



$$\operatorname{div}(\vec{F}) = (4z + xz + 3) \rightarrow 4z + rz\cos\theta + 3$$

$$\int_0^4 \int_0^{\pi} \int_r^4 (4z + 3 + zr\cos\theta)\, r\, dr\, d\theta\, dz \qquad zr\cos\theta \rightarrow 0$$

$$3\int_0^4 \int_0^{\pi} \int_r^4 r\, dr\, d\theta\, dz + 4\int_0^4 \int_0^{\pi} \int_r^4 z\, r\, dr\, d\theta\, dz$$

$$3\pi \int_0^4 \int_r^4 r\, dr\, dz + 4\pi \int_0^4 \int_r^4 z\, r\, dr\, dz$$

$$3\pi \int_0^4 r(4-r)\, dr + 4\pi \int_0^4 \left(\frac{4^2}{2} - \frac{r^2}{2}\right) r\, dr$$

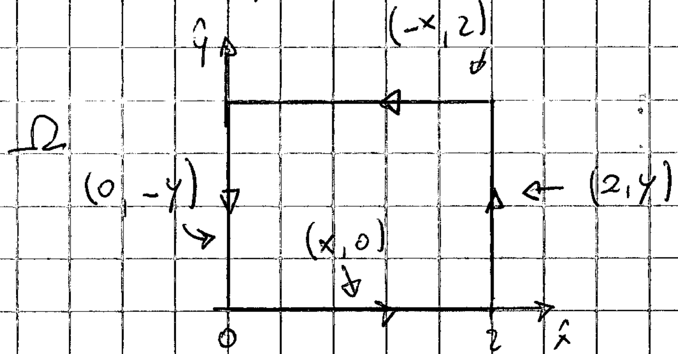
$$3\pi \left\{ 4\frac{4^2}{2} - \frac{4^3}{3} \right\}$$

$$3\pi \left\{ \frac{4^3}{6} \right\} + \frac{4\pi}{2} \left\{ \frac{4^2 \cdot 4^2}{2} - \frac{4^4}{4} \right\}$$

$$\frac{\pi 4^3}{2} + 2\pi\left(\frac{4^4}{4}\right) = \boxed{160\pi}$$

P3) En efecto,

$$\vec{F} = (y^2, x)$$



$$\oint_{\Gamma} \vec{F} \cdot d\vec{l} = \underbrace{\oint_{\Gamma_1} \vec{F} \cdot d\vec{l}_1}_{(1)} + \underbrace{\oint_{\Gamma_2} \vec{F} \cdot d\vec{l}_2}_{(2)} + \underbrace{\oint_{\Gamma_3} \vec{F} \cdot d\vec{l}_3}_{(3)} + \underbrace{\oint_{\Gamma_4} \vec{F} \cdot d\vec{l}_4}_{(4)}$$

$$\vec{r}_1 = x\hat{x} \quad , \quad \vec{r}_2 = y\hat{y} \quad , \quad \vec{r}_3 = -x\hat{x} \quad , \quad \vec{r}_4 = -y\hat{y}$$

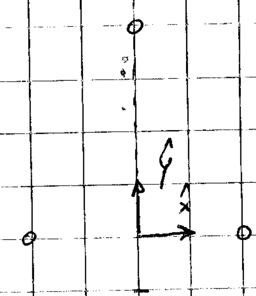
$$(1) : \int_0^2 0\, dx + \int_0^2 2\, dy + \int_2^0 -2^2\, dx$$

$$+ \int_2^0 0 \cdot dy = \boxed{12}.$$

P4)

Usaremos las siguientes ecuaciones.

$$\vec{F} = m \cdot \vec{a}.$$



$$\vec{F} = q\,\vec{E}$$

$$q E = m \frac{d^2x}{dt^2}$$

$$x(t) = \frac{q}{m} E \frac{t^2}{2} + c_1 t + c_2$$

$$\dot{x}(t) = \frac{qE}{m} t + c_1$$

Condiciones iniciales $x(t=0) = 0$
$\dot{x}(t=0) = 0$

$$\Rightarrow \quad c_1 = c_2 = 0$$

$$x(t) = \frac{q}{m} E \frac{t^2}{2}$$

además, Tomamos caída libre.

$$g = \frac{d^2y}{dt^2} \rightarrow y(t) = \frac{gt^2}{2}$$

$$h = \frac{gt^2}{2} \rightarrow t = \sqrt{\frac{2h}{g}}$$

$$\Rightarrow \boxed{\begin{aligned} x &= 0.3670 \, m \\ 2x &= 73.47 \, cm \end{aligned}}$$

P5)

$\vec{r} = z\hat{k}$

$\vec{r}' = R\hat{r}$

$\theta \in [0, 2\pi)$

$$\lambda = \frac{Q}{2\pi R}$$

$$d\vec{l} = \cancel{R\cos\theta}\ R\, d\theta$$

$$\vec{E}(z) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{2\pi \cancel{R}} \int_0^{2\pi} \frac{(z\hat{k} - R\hat{r})}{(z^2+R^2)^{3/2}} \cancel{R}\, d\theta$$

$$\vec{E}(z) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{\cancel{2\pi}} \cancel{2\pi} \left\} \frac{z\hat{k}}{(z^2+R^2)^{3/2}} \right\{$$

$$\vec{E}(z) = \frac{Q}{4\pi\varepsilon_0} \left\} \frac{z\hat{k}}{(z^2+R^2)^{3/2}} \right\{$$

P6)

a) $\vec{r} = b\hat{k}$ $\quad \theta \in (0, 2\pi)$ $\quad \sigma = \dfrac{Q}{\pi a^2}$

$\vec{r}' = r\hat{r}$ $\quad r \in (0, a)$;

$$\vec{E}(b) = \int_0^a \int_0^{2\pi} \frac{1}{4\pi\varepsilon_0} \frac{Q}{\pi a^2} \left(\frac{b\hat{k} - r\hat{r}}{(b^2 + r^2)^{3/2}} \right) r\,dr\,d\theta .$$

$$\vec{E}(b) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{\pi a^2} \int_0^a \int_0^{2\pi} \frac{br\,dr\,d\theta\,\hat{k}}{(b^2 + r^2)^{3/2}}$$

$$\vec{E}(b) = \frac{1}{\cancel{4\pi}\varepsilon_0} \frac{Q}{\pi a^2} \cancel{2\pi} \int_0^a \frac{br\,dr}{(b^2 + r^2)^{3/2}}$$

$b^2 + r^2 = u$

$2r\,dr = du$

$$\vec{E}(b) = \frac{bQ}{2\pi \varepsilon_0 a^2} \int_{b^2}^{a^2+b^2} \frac{\cancel{r}}{u^{3/2}} \frac{du}{2\cancel{r}} \hat{k}$$

$$\vec{E}(b) = \frac{bQ}{4\pi \varepsilon_0 a^2} \int_{b^2}^{a^2+b^2} u^{-3/2} \, du \, \hat{k}$$

$$\vec{E}(b) = \frac{bQ}{4\pi \varepsilon_0 a^2} \left\{ \left. \frac{u^{-1/2}}{-1/2} \right|_{b^2}^{a^2+b^2} \right\} \hat{k}$$

$$\boxed{\vec{E}(b) = \frac{bQ}{2\pi \varepsilon_0 a^2} \left\{ \frac{1}{\sqrt{b^2}} - \frac{1}{\sqrt{a^2+b^2}} \right\} \hat{k}}$$

b) $$\lim_{a \to \infty} \vec{E}(b) = \frac{bQ}{2\pi \varepsilon_0 a^2} \frac{1}{|b|}$$

$$\sigma = \frac{Q}{\pi a^2}$$

$$\vec{E}(b) = \frac{\sigma b}{2\varepsilon_0 |b|} = \boxed{\frac{\sigma}{2\varepsilon_0} \operatorname{Sg}(b)}$$

e) $\vec{F} = Q\vec{E}$

$d\vec{F} = dQ\,\vec{E}$, $dQ = \lambda\, dz$.

$d\vec{F} = \lambda\, dz\, \vec{E}$

$\vec{F} = \int \lambda\, dz\, \vec{E}(z)$

$$\vec{F} = \frac{Q}{2\pi\varepsilon_0 a^2}\left\{\int_b^{b+L} dz - \int_b^{b+L}\frac{z}{\sqrt{a^2+z^2}}\,dz\right\}\hat{k} \quad \text{CV}$$

$u = a^2+z^2$

$$\vec{F} = \frac{Q}{2\pi\varepsilon_0 a^2}\left\{L - \int_{a^2+b^2}^{a^2+(b+L)^2}\frac{du}{2u^{1/2}}\right\}\hat{k} \qquad du = 2z\,dz$$

$$\vec{F} = \frac{Q}{2\pi\varepsilon_0 a^2}\left\{L - \frac{u^{1/2}}{\cancel{2}\,\cancel{1/2}}\Bigg|_{a^2+b^2}^{a^2+(b+L)^2}\right\}$$

$$\boxed{\vec{F} = \frac{Q}{2\pi\varepsilon_0 a^2}\left(L - \sqrt{a^2+(b+L)^2} - \sqrt{a^2+b^2}\right)\hat{k}}$$