Introducción al Álgebra 2018
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Pauta Guía Sumatorias

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(**P**1) (a)

$$\sum_{k=1}^{n-1} k(k+1) = \sum_{k=1}^{n-1}(k^2+k) \tag{1}$$
$$= \sum_{k=1}^{n-1} k^2 + \sum_{k=1}^{n-1} k \tag{2}$$
$$= \frac{(n-1)n(2(n-1)+1)}{6} + \frac{(n-1)n}{2} \tag{3}$$
$$= \frac{(n-1)n(2n-1)}{6} + \frac{(n-1)n}{2} \tag{4}$$

(b)

$$\sum_{k=1}^{n-1}(k+nk)(k-n) = \sum_{k=1}^{n-1} k^2 - n\sum_{k=1}^{n-1} k + n\sum_{k=1}^{n-1} k^2 - n^2\sum_{k=1}^{n-1} k \tag{5}$$
$$= \frac{(n-1)n(2n-1)}{6} - n\cdot\frac{(n-1)n}{2} + n\cdot\frac{(n-1)n(2n-1)}{6} - n^2\cdot\frac{(n-1)n}{2} \tag{6}$$

(c)

$$\sum_{k=1}^{n} k\ln\left(1+\frac{1}{k}\right) = \sum_{k=1}^{n} k\ln\left(\frac{k+1}{k}\right) = \sum_{k=1}^{n} k(\ln(k+1)-\ln(k)) \tag{7}$$
$$= \sum_{k=1}^{n} k\ln(k+1) - k\ln(k) \tag{8}$$
$$= \sum_{k=1}^{n}(k+1-1)\ln(k+1) - k\ln(k) \tag{9}$$
$$= \sum_{k=1}^{n}[(k+1)\ln(k+1) - k\ln(k)] - \sum_{k=1}^{n}\ln(k+1) \tag{10}$$
$$= (n+1)\ln(n+1) - \ln(1) - \ln\left(\prod_{k=1}^{n}(k+1)\right) \tag{11}$$
$$= (n+1)\ln(n+1) - \ln((n+1)!) \tag{12}$$

(d)

$$\sum_{i=0}^{n}\sum_{k=i+1}^{n} i^2 = \sum_{i=0}^{n}(n-(i+1)+1)i^2 \tag{13}$$
$$= \sum_{i=0}^{n}(n-i)i^2 = \sum_{i=0}^{n} ni^2 - \sum_{i=0}^{n} i^3 \tag{14}$$
$$= n\cdot\frac{n(n+1)(2n+1)}{6} - \left(\frac{n(n+1)}{2}\right)^2 \tag{15}$$

(e)

$$\sum_{i=0}^{n}\sum_{k=i+1}^{n} 4 = \sum_{i=0}^{n} 4(n-i) \tag{16}$$
$$= 4\sum_{i=0}^{n} n - 4\sum_{i=0}^{n} i \tag{17}$$
$$= (n-0+1)4n - \frac{4n(n+1)}{2} = 4n(n+1) - 2n(n+1) \tag{18}$$
$$= 2n(n+1) \tag{19}$$

(**P**2) En efecto, si $x = y$ es trivial, ahora suponemos $x \neq y$:

$$\sum_{i=0}^{n-1}(x-y)x^{n-i-1}y^i = \sum_{i=0}^{n-1} x^{n-i}y^i - x^{n-(i+1)}y^{i+1} \tag{20}$$
$$= x^n y^0 - x^0 y^n = x^n - y^n \tag{21}$$

luego, pasamos dividiendo y concluimos.

(**P**3) Procederemos por inducción. Primero veamos la primera desigualdad:
Caso base: $n = 1$
$1 + \frac{n}{2} = 1 + \frac{1}{2}$,
$H_{2^n} = H_2 = 1 + \frac{1}{2}$,
$\Rightarrow 1 + \frac{n}{2} \leq H_{2^n}$
Hipótesis inductiva: Sea k tal que

$$1 + \frac{k}{2} \leq H_{2^k}$$

PDQ $1 + \frac{k+1}{2} \leq H_{2^{k+1}}$
En efecto,
$H_{2^{k+1}} = H_{2^k} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^{k+1}}$, esto veanlo bien!
notar que a H_{2^k} se le suman 2^k términos, todos mayores o iguales a $\frac{1}{2^{k+1}}$, por lo tanto:

$$H_{2^k} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^{k+1}} \geq H_{2^k} + 2^k\cdot\frac{1}{2^{k+1}} \tag{22}$$
$$\geq 1 + \frac{k}{2} + \frac{1}{2} = 1 + \frac{k+1}{2} \tag{23}$$

donde la última desigualdad se desprende de la hipótesis inductiva.
Para la segunda desigualdad:
Caso base: $n = 1$
$1 + n = 1 + 1 = 2$,
$H_{2^n} = H_2 = 1 + \frac{1}{2}$,
$\Rightarrow H_{2^n} \leq 1 + n$
Hipótesis inductiva: Sea k tal que

$$H_{2^k} \leq 1 + k$$

PDQ $H_{2^{k+1}} \leq 1 + k + 1$
análogamente a la desigualdad anterior, notar que a H_{2^k} se le suman 2^k términos, todos menores o iguales a $\frac{1}{2^k}$, por lo tanto:

$$\begin{aligned} H_{2^k} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^{k+1}} &\leq H_{2^k} + 2^k \cdot \frac{1}{2^k} && (24) \\ &\leq 1 + k + 1 && (25) \end{aligned}$$

(**P**4)

$$\begin{aligned} \sum_{i=\frac{m(m-1)}{2}+1}^{\frac{m(m+1)}{2}} 2i - 1 &= \sum_{i=1}^{\frac{m(m+1)}{2}-\frac{m(m-1)}{2}} 2\left(i + \frac{m(m-1)}{2}\right) - 1 && (26) \\ &= \sum_{i=1}^{m} 2\left(i + \frac{m(m-1)}{2}\right) - 1 && (27) \\ &= 2\sum_{i=1}^{m} i + 2\sum_{i=1}^{m} \frac{m(m-1)}{2} - \sum_{i=1}^{m} 1 && (28) \\ &= 2 \cdot \frac{m(m+1)}{2} + 2(m-1+1)\frac{m(m-1)}{2} - (m-1+1) && (29) \\ &= m(m+1) + m^2(m-1) - m && (30) \\ &= m(m+1+m(m-1)-1) && (31) \\ &= m(m+1+m^2-m-1) && (32) \\ &= m^3 && (33) \end{aligned}$$

(**P**5) Ya en el foro

(**P**6) *a*) Trabajamos con fracciones parciales:
$\frac{1}{k(k-1)} = \frac{A}{k} + \frac{B}{k-1}$
$\Rightarrow 1 = A(k-1) + Bk = (A+B)k - A$
$\Rightarrow A = -1,\ B = 1 \Rightarrow \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$ Luego, tenemos que

$$\sum_{k=2}^{n} \frac{1}{k(k-1)} = \sum_{k=2}^{n} \frac{1}{k-1} - \frac{1}{k} = 1 - \frac{1}{n}$$

b)

$$\sum_{k=1}^{n} \frac{2k+1}{k^2(k+1)^2} = \sum_{k=1}^{n} \frac{k^2+2k+1-k^2}{k^2(k+1)^2} \tag{34}$$

$$= \sum_{k=1}^{n} \frac{(k+1)^2-k^2}{k^2(k+1)^2} \tag{35}$$

$$= \sum_{k=1}^{n} \frac{1}{k^2} - \frac{1}{(k+1)^2} \tag{36}$$

$$= 1 - \frac{1}{(n+1)^2} \tag{37}$$

c)

$$\sum_{i=5}^{n} \sum_{j=1}^{i} \frac{i+1}{j(j+1)} = \sum_{i=5}^{n} (i+1) \sum_{j=1}^{i} \frac{1}{j(j+1)} \tag{38}$$

$$= \sum_{i=5}^{n} (i+1) \sum_{j=1}^{i} \frac{1}{j} - \frac{1}{j+1} \tag{39}$$

$$= \sum_{i=5}^{n} (i+1) \left(1 - \frac{1}{i+1}\right) \tag{40}$$

$$= \sum_{i=5}^{n} (i+1) - 1 = \sum_{i=5}^{n} i \tag{41}$$

$$= \sum_{i=1}^{n} i - 4 - 3 - 2 - 1 \tag{42}$$

$$= \frac{n(n+1)}{2} - 10 \tag{43}$$

(**P7**) *a*)

$$\sum_{k=1}^{n} k \cdot k! = \sum_{k=1}^{n} (k+1-1) \cdot k! \tag{44}$$

$$= \sum_{k=1}^{n} (k+1)k! - k! \tag{45}$$

$$= \sum_{k=1}^{n} (k+1)! - k! \tag{46}$$

$$= (n+1)! - 1 \tag{47}$$

b)

$$\sum_{k\ impar}^{n} \frac{1}{\sqrt[3]{(k+1)^2} + \sqrt[3]{(k+1)(k-1)} + \sqrt[3]{(k-1)^2}} \tag{48}$$

$$= \sum_{k\ impar}^{n} \frac{1}{\sqrt[3]{(k+1)^2} + \sqrt[3]{(k+1)(k-1)} + \sqrt[3]{(k-1)^2}} \cdot \frac{\sqrt[3]{(k+1)^2} - \sqrt[3]{(k-1)^2}}{\sqrt[3]{(k+1)^2} - \sqrt[3]{(k-1)^2}} \tag{49}$$

$$= \sum_{k\ impar}^{n} \frac{\sqrt[3]{(k+1)^2} - \sqrt[3]{(k-1)^2}}{2} \tag{50}$$

$$= \frac{\sqrt[3]{(n+1)^2}}{2} \tag{51}$$

c)

$$\sum_{k=1}^{n} \frac{k2^k}{(k+2)!} = \sum_{k=1}^{n} \frac{(k+2-2)2^k}{(k+2)!} \tag{52}$$

$$= \sum_{k=1}^{n} \frac{(k+2)2^k}{(k+2)!} - \frac{2^{k+1}}{(k+2)!} \tag{53}$$

$$= \sum_{k=1}^{n} \frac{(k+2)2^k}{(k+2)!} - \frac{(k+3)2^{k+1}}{(k+3)!} \tag{54}$$

$$= \frac{(1+2)2^1}{(1+2)!} - \frac{(n+3)2^{n+1}}{(n+3)!} \tag{55}$$

$$= 1 - \frac{2^{n+1}}{(n+2)!} \tag{56}$$

(**P**8) *a*) Términos pares:

$$\sum_{i=1}^{n} (-1)^{2i}(2i)^2 = \sum_{i=1}^{n} 4i^2 \tag{57}$$

$$= 4\sum_{i=1}^{n} i^2 \tag{58}$$

$$= 4 \cdot \frac{n(n+1)(2n+1)}{6} \tag{59}$$

b) Términos impares:

$$\sum_{i=1}^{n} (-1)^{2i-1}(2i-1)^2 = -\sum_{i=1}^{n} 4i^2 - 4i + 1 \tag{60}$$

$$= -4\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 \tag{61}$$

$$= -4 \cdot \frac{n(n+1)(2n+1)}{6} + 4 \cdot \frac{n(n+1)}{2} - n \tag{62}$$

Por lo tanto, al sumar ambas partes, queda:

$$\sum_{k=1}^{2n}(-1)^k k^2 = 2n(n+1) - n$$

(**P**9) *a*)

$$\begin{aligned}
a + aa + aaa + ... + aa...aa &= a(1 + 11 + 111 + ... + 11..,11) && (63)\\
&= a(1 + (1+10) + (1+10+100) + ... + (1+10+...+10^{n-1})) && (64)\\
&= a\left(\sum_{k=0}^{n-1}\sum_{i=0}^{k} 10^i\right) && (65)\\
&= a\left(\sum_{k=0}^{n-1}\frac{10^{i+1}-1}{10-1}\right) && (66)\\
&= \frac{a}{9}\left(\sum_{k=0}^{n-1}10^{i+1} - \sum_{k=0}^{n-1}1\right) && (67)\\
&= \frac{a}{9}\left(\frac{10^n-10}{10-1} - (n-1-0+1)1\right) && (68)\\
&= \frac{a}{9}\left(\frac{10^n-10}{9} - n\right) && (69)
\end{aligned}$$

b) Tenemos que la suma $1 + \frac{1+2}{2} + \frac{1+2+3}{3} + ... + \frac{1+2+...+n}{n}$ se puede escribir como:

$$\begin{aligned}
\sum_{k=1}^{n}\sum_{i=1}^{k}\frac{i}{k} &= \sum_{k=1}^{n}\frac{1}{k}\sum_{i=1}^{k} i && (70)\\
&= \sum_{k=1}^{n}\frac{1}{k}\cdot\frac{k(k+1)}{2} && (71)\\
&= \frac{1}{2}\left(\sum_{k=1}^{n}k + \sum_{k=1}^{n}1\right) && (72)\\
&= \frac{1}{2}\left(\frac{n(n+1)}{2} + n\right) && (73)
\end{aligned}$$

(**P**10)

$$\sum_{k=0}^{n} k2^k + (n+1)2^{n+1} = \sum_{k=0}^{n}(k+1)2^{k+1} \tag{74}$$

$$= \sum_{k=0}^{n}(k+1)2 \cdot 2^k \tag{75}$$

$$= 2\sum_{k=0}^{n} k2^k + 2\sum_{k=0}^{n} 2^k \tag{76}$$

$$= 2\sum_{k=0}^{n} k2^k + 2\left(\frac{2^{n+1}-1}{2-1}\right) \tag{77}$$

$$\Rightarrow (n+1)2^{n+1} - 2(2^{n+1}-1) = \sum_{k=0}^{n} k2^k \tag{78}$$

$$\Rightarrow \sum_{k=0}^{n} k2^k = 2(n2^n - 2^n + 1) \tag{79}$$

(**P**11) en proceso

(**P**12)

$$
\begin{aligned}
S_n &= \sum_{k=1}^{n} kr^k && (80)\\
&= \sum_{k=0}^{n-1}(k+1)r^{k+1} && (81)\\
&= \sum_{k=0}^{n-1} kr^{k+1} + \sum_{k=0}^{n-1} r^{k+1} && (82)\\
&= r\sum_{k=0}^{n-1} kr^k + \sum_{k=0}^{n-1} r^{k+1} && (83)\\
1 &= r\left(\sum_{k=0}^{n} kr^k - nr^n\right) + \sum_{k=0}^{n-1} r^{k+1} && (84)\\
&= r(S_n - nr^n) + \sum_{k=0}^{n-1} r^{k+1} \text{ Primera parte} && (85)\\
S_n - rS_n &= -nr^{n+1} + r\sum_{k=0}^{n-1} r^k && (86)\\
S_n(1-r) &= -nr^{n+1} + r\left(\frac{1-r^n}{1-r}\right) && (87)\\
S_n &= \frac{-nr^{n+1}}{1-r} + \frac{r-r^{n+1}}{(1-r)^2} && (88)\\
&= \frac{-nr^{n+1}(1-r) + r - r^{n+1}}{(1-r)^2} && (89)\\
&= \frac{-nr^{n+1} + nr^{n+2} + r - r^{n+1}}{(1-r)^2} && (90)\\
&= \frac{r-(n+1)r^{n+1} + nr^{n+2}}{(1-r)^2} && (91)
\end{aligned}
$$

(**P**13) en proceso

(**P**14)

$$\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \frac{1}{2}\sum_{k=1}^{n} \frac{2}{k(k+1)(k+2)} \tag{92}$$

$$= \frac{1}{2}\sum_{k=1}^{n} \frac{k+2-k}{k(k+1)(k+2)} \tag{93}$$

$$= \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)} \tag{94}$$

$$= \frac{1}{2}\left(\frac{1}{1\cdot(1+1)} - \frac{1}{(n+1)(n+2)}\right) \tag{95}$$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right) \tag{96}$$

(**P**15) Notamos que la suma de los n números consecutivos a partir de k_0 termina en el número k_0+n-1, por lo tanto, es igual a:

$$\sum_{i=k_0}^{k_0+n-1} i = \sum_{k=1}^{n} i + k_0 - 1 \tag{97}$$

$$= \sum_{k=1}^{n} i + \sum_{k=1}^{n} k_0 - \sum_{k=1}^{n} 1 \tag{98}$$

$$= \frac{n(n+1)}{2} + n\cdot k_0 - n \tag{99}$$

Como n es impar, tenemos que $n+1$ es par, por lo tanto, divisible por 2, definimos $z \in \mathbb{Z}$ tal que $z = \dfrac{n+1}{2}$ y nos queda

$$\sum_{i=k_0}^{k_0+n-1} i = nz + nk_0 - n$$

lo que es divisible por n

(**P**16) Separando el caso de los pares con el de los impares, tenemos que $n/2$ es entero, por lo tanto, se puede comenzar por los pares:

$$\sum_{i=1}^{n/2} (-1)^{2i+1}\frac{1}{4^{2i}} = -\sum_{i=1}^{n/2} \frac{1}{16^i} \tag{100}$$

$$= -\left(\frac{\frac{1}{16} - \left(\frac{1}{16}\right)^{n/2+1}}{1-\frac{1}{16}}\right) \tag{101}$$

luego, los impares:

$$\sum_{i=1}^{n/2}(-1)^{2i-1+1}\frac{1}{4^{2i-1}} = \sum_{i=1}^{n/2}\frac{1}{4^{2i-1}} \tag{102}$$

$$= 4\sum_{i=1}^{n/2}\frac{1}{16^i} \tag{103}$$

$$= 4\left(\frac{\frac{1}{16}-\left(\frac{1}{16}\right)^{n/2+1}}{1-\frac{1}{16}}\right) \tag{104}$$

sumando todo, queda:

$$\sum_{k=0}^{n}(-1)^{k+1}\frac{1}{4^k} = 4\left(\frac{\frac{1}{16}-\left(\frac{1}{16}\right)^{n/2+1}}{1-\frac{1}{16}}\right) - \left(\frac{\frac{1}{16}-\left(\frac{1}{16}\right)^{n/2+1}}{1-\frac{1}{16}}\right) \tag{105}$$

$$= 3\left(\frac{\frac{1}{16}-\left(\frac{1}{16}\right)^{n/2+1}}{1-\frac{1}{16}}\right) \tag{106}$$

$$= 3\left(\frac{\frac{1}{16}-\left(\frac{1}{16}\right)^{n/2+1}}{\frac{15}{16}}\right) \tag{107}$$

$$= \frac{3\cdot 16}{15}\left(\frac{1}{16}-\left(\frac{1}{16}\right)^{n/2+1}\right) \tag{108}$$

$$= \frac{1}{5}\left(\frac{16}{16}-\left(\frac{1}{16}\right)^{n/2+1-1}\right) \tag{109}$$

$$= \frac{1}{5}\left(1-\left(\frac{1}{16}\right)^{n/2}\right) \tag{110}$$

$$= \frac{1}{5}\left(1-\left(\frac{1}{4}\right)^{n}\right) \tag{111}$$

$$= \frac{1}{5}\left(1-\frac{1}{4^n}\right) \tag{112}$$