Auxiliar 4

P1

Sea $\varphi \in \mathcal{D}(\mathbb{R})$ no idénticamente nula. Pruebe que $\hat{\varphi} \notin \mathcal{D}(\mathbb{R})$.

Dem:

Usamos: Liouville

Toda función entera y acotada es constante.

- Probemos que $\hat{\varphi}$ es acotada:

$$|\hat{\varphi}(k)| \leq \frac{1}{(2\pi)^{1/2}} \int_{sop(\varphi)} |\varphi(x)| \, |e^{-ix\cdot k}| \, dx$$

$$\leq \frac{1}{(2\pi)^{1/2}} \|\varphi\|_\infty \, |sop(\varphi)| < \infty.$$

- Probemos φ entera. Sup. $sop(\varphi) \subset B(0,R)$

$$\hat{\varphi}(k) = \frac{1}{\sqrt{2\pi}} \int_{B(0,R)} \varphi(x) e^{-ix\cdot k} \, dk = \frac{1}{\sqrt{2\pi}} \int_{B(0,R)} \varphi(x) \sum_{k=0}^{\infty} \frac{(-ik\cdot x)^z}{z!} \, dx$$

$$e^z = \sum_{z=0}^{\infty} \frac{x^z}{z!}$$

Notemos que $\sum_{n=0}^{\infty} \frac{(-i k\cdot x)^n}{n!}$ converge uniformemente cuando $|x\cdot k| < r$, $\forall r>0$. En particular, como k es fijo, converge unif. $\forall x$ tal que $|x| < R$. Por lo tanto

$$\hat{\varphi}(k) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \int_{B(0,R)} \varphi(x) \frac{(-i k\cdot x)^n}{n!} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} k^n \frac{(-i)^n}{n!} \int_{B(0,R)} \varphi(x)\, x^n \, dx$$

Notemos que la serie es convergente para todo k con radio de convergencia infinito.

$\Rightarrow$ es entera.

Así, por Liouville, $\hat{\varphi}$ es constante.

Supongamos, por absurdo, $\hat{\varphi}$ tiene soporte compacto, $\hat{\varphi} \neq 0$

$$\Rightarrow \hat{\varphi}\big|_{sop(\hat{\varphi})^c} = 0 \quad \underset{\text{por ser constante}}{\Rightarrow} \quad \hat{\varphi} \equiv 0 \;!!$$

P2|

(a). Sea $\alpha \in (0,\infty)$. Comprobar que

$$\mathcal{F}\left(\exp(-\alpha|x|^2)\right)(k) = \frac{1}{(2\alpha)^{d/2}} \exp\left(-\frac{|k|^2}{4\alpha}\right)$$

Dem:

$$\mathcal{F}\left(\exp(-\alpha|x|^2)\right)(k) = \mathcal{F}\left(\exp\left(-2\alpha\frac{|x|^2}{2}\right)\right)(k)$$

$$= \mathcal{F}\left(\sigma_{\sqrt{2\alpha}} \exp\left(-\frac{|x|^2}{2}\right)\right)(k) = \frac{1}{(2\alpha)^{d/2}} \mathcal{F}\left(\exp\left(-\frac{|x|^2}{2}\right)\right)\left(\frac{k}{\sqrt{2\alpha}}\right)$$

$\downarrow$

$$\mathcal{F}(\sigma_\lambda u) = |\lambda|^{-d}\left(\sigma_{\frac{1}{\lambda}} \mathcal{F}(u)\right)$$

$$= \frac{1}{(2\alpha)^{d/2}} \exp\left(-\alpha|k|^2\right)$$

$\downarrow$

Vimos en clase que $\mathcal{F}\left(\exp(-|x|^2/2)\right) = \exp(-|k|^2/2)$

(b). Demostrar el siguiente resultado:

Prop: Sea $d \geq 3$, $C_\alpha = 2^{d/2}\,\Gamma\left(\frac{\alpha}{2}\right)$ donde $0<\alpha<d$ y

$$\Gamma(\beta) = \int_0^\infty e^{-t}\, t^{\beta-1}\, dt$$

$$\Rightarrow \mathcal{F}^*\left(C_\alpha T_{\frac{1}{|k|^\alpha}}\right)(x) = C_{d-\alpha}\, T_{\frac{1}{|x|^{d-\alpha}}}.$$

Dem:

Notar que $\frac{1}{|x|^\alpha} \in L^1_{loc}$ para $0<\alpha<d$ y está acotada $\left(\text{prob. aux}\right)$

$$\Rightarrow T_{\frac{1}{|x|^\alpha}} \in S'(\mathbb{R}^d)$$

Por lo tanto, no hay problemas para tomar Fourier

Claim:

$$\int_0^\infty e^{-\frac{|x|^2}{2}\lambda} \lambda^{\frac{\alpha}{2}-1} d\lambda = \frac{C_\alpha}{|x|^\alpha} \quad (*_1)$$

En efecto,

$$\int_0^\infty e^{-\frac{|x|^2}{2}\lambda} \lambda^{\frac{\alpha}{2}-1} d\lambda = \frac{2}{|x|^2}\int_0^\infty e^{-t}\, t^{\frac{\alpha}{2}-1} dt$$

$$t = \frac{|x|^2\lambda}{2} \Rightarrow \lambda = \frac{2t}{|x|^2}, \quad \lambda^{\frac{\alpha}{2}-1} = \left(\frac{2t}{|x|^2}\right)^{\frac{\alpha}{2}-1}$$

$$d\lambda = \frac{2}{|x|^2} dt \Rightarrow \lambda^{\frac{\alpha}{2}-1} d\lambda = \left(\frac{2}{|x|^2}\right)^{\frac{\alpha}{2}} t^{\frac{\alpha}{2}-1} dt$$

$$= \frac{2^{\alpha/2}}{|x|^\alpha}\Gamma\left(\frac{\alpha}{2}\right) = \frac{C_\alpha}{|x|^\alpha}$$

Sea $\varphi \in \mathcal{D}(\mathbb{R}^d_x)$

$$\mathcal{F}^*\left(C_\alpha T_{\frac{1}{|k|^\alpha}}\right)[\varphi] \underset{(\text{def})}{=} C_\alpha T_{\frac{1}{|k|^\alpha}}[\mathcal{F}(\varphi)]$$

$$= C_\alpha \int_{\mathbb{R}^d_k} \frac{1}{|k|^\alpha}\mathcal{F}(\varphi)(k)\, dk$$

$$= \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d_k} C_\alpha \frac{1}{|k|^\alpha}\int_{\mathbb{R}^d_x} \varphi(x) e^{-ik\cdot x} dx\, dk$$

$$\underset{\substack{\downarrow\\ \text{por } *1}}{=} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d_k} \left(\int_0^\infty e^{-\frac{|k|^2\lambda}{2}} \lambda^{\frac{\alpha}{2}-1} d\lambda \right) \left(\int_{\mathbb{R}^d_x} e^{-ix\cdot k} \varphi(x)\, dx \right) dk$$

$$\underset{(\text{Fubini})}{=} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d_x} \varphi(x) \int_0^\infty \lambda^{\frac{\alpha}{2}-1} \int_{\mathbb{R}^d_k} e^{-ix\cdot k} e^{-\frac{|k|^2\lambda}{2}} \, dk\, d\lambda\, dx$$

$$= \int_{\mathbb{R}^d_x} \varphi(x) \int_0^\infty \lambda^{\frac{\alpha}{2}-1} \left(\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d_k} e^{-ix\cdot k} e^{-\frac{|k|^2\lambda}{2}} dk \right) d\lambda\, dx$$

$$= \int_{\mathbb{R}^d_x} \varphi(x) \int_0^\infty \lambda^{\frac{\alpha}{2}-1} \mathcal{F}\left(\exp\left(-\frac{\lambda}{2}|k|^2\right)\right) d\lambda\, dk$$

$$\underset{\substack{\downarrow\\ \text{por (a) con}}}{=} \int_{\mathbb{R}^d_x} \varphi(x) \int_0^\infty \lambda^{\frac{\alpha}{2}-\frac{d}{2}-1} e^{-\frac{|x|^2}{2\lambda}} d\lambda\, dx$$

$$\alpha = \frac{\lambda}{2} \Rightarrow \frac{1}{(2\alpha)^{d/2}} \exp\left(-\frac{|x|^2}{4\alpha}\right) = \frac{1}{(\lambda)^{d/2}} \exp\left(-\frac{|x|^2}{2\lambda}\right)$$

$$\underset{\downarrow}{=} \int_{\mathbb{R}^d_x} \varphi(x) \int_0^\infty \mu^{\frac{d}{2}-\frac{\alpha}{2}-1} e^{-\frac{\mu|x|^2}{2}} d\mu\, dx$$

$$\mu = \frac{1}{\lambda} \Rightarrow d\mu = -\frac{1}{\lambda^2} d\lambda$$

$$\Rightarrow \frac{d\mu}{\mu} = -\frac{d\lambda}{\lambda}$$ pero el menos se va al cambiar los límites de integración.

$$\underset{\text{por } (*_1)}{=} \int_{\mathbb{R}^d_x} \varphi(x) \frac{C_{d-\alpha}}{|x|^{d-\alpha}} dx = C^{d-\alpha} \, T_{\frac{1}{|x|^{d-\alpha}}} [\varphi].$$

(c). Consideremos la ecuación de Poisson:

sea $f \in S(\mathbb{R}^d)$, $u \in W^{1,2}(\mathbb{R}^d)$,

$$-\Delta u = f.$$

Encontrar por Fourier la solución de Poisson para $d \geq 3$.

Por Fourier, $|k|^2 \hat{u} = \hat{f}$ (como $f \in S(\mathbb{R}^d)$ y $u \in L^2(\mathbb{R}^d)$ no hay problemas al tomar Fourier)

$$\Rightarrow \hat{u} = \frac{\hat{f}}{|k|^2}.$$

$$\Rightarrow F^*(F(u))(x) = F^*\left(\hat{f} \cdot \frac{1}{|k|^2}\right)(x) = F^*(\hat{f}(x)) * F^*\left(\frac{1}{|k|^2}\right)(x)$$

Fourier lleva convolución a producto.

$$\Rightarrow u(x) = f(x) * F^*\left(\frac{1}{|k|^2}\right)(x).$$

$F: L^2 \to L^2$ es isometría.

$$F^*(F(u)) = u \quad \forall u \in L^2(\mathbb{R}^d).$$

Como F^* es lineal, por P2.1 tenemos que

$$C_2 \, F^*\left(\frac{1}{|k|^2}\right)(x) = C_{d-2} \frac{1}{|x|^{d-2}}$$

$$\Rightarrow \mathcal{F}^*\left(\frac{1}{|k|^2}\right) = \frac{C_{d-2}}{C_2} \frac{1}{|x|^{d-2}}$$

$$\Rightarrow u(x) = \frac{C_{d-2}}{C_2}\left(f * \frac{1}{|\cdot|^{d-2}} \right)(x).$$

Notar que, como $\frac{1}{|x|^{d-2}} \in D(\mathbb{R}^d)$ y $f \in S(\mathbb{R}^d)$

$\Rightarrow u \in S(\mathbb{R}^d)$. $\Rightarrow$ ESTAMOS BIEN

¿Qué pasa si $f \in L^p(\mathbb{R}^d)$, $p > 1$?

No podemos asegurar mucho sobre u como para justificar los cuentos de antes.

Por lo tanto, necesitaríamos Hardy-Littlewood-Sobolev:

TEOREMA:

$p, q > 1$, $0 < \lambda < d$, $\frac{1}{p} + \frac{\lambda}{d} + \frac{1}{q} = 2$, $f \in L^p$, $g \in L^q$.

$$\Rightarrow \iint \left(\frac{f(x)\, g(y)}{|x-y|} \right) dx\, dy \leq C_{d,\lambda,q} \, \|f\|_{L^p} \|g\|_{L^q}$$

Con H-L-S tenemos la regularidad que necesitamos para u.

P3) Encontrar solución de la ecuación

$$\begin{cases} u_t - \Delta u = 0 & \text{en } \mathbb{R}^d \times (0,\infty) \\ u(x,0) = f(x) & x \in \mathbb{R}^d, \; f \in S(\mathbb{R}^d) \end{cases}$$

sup $u \in W^{1,2}(\mathbb{R}^d)$

$$\Rightarrow \hat{u} = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} u(x,t) e^{-ix\cdot k} \, dx$$

$$\Rightarrow \hat{u_t} = (\hat{u})_t \quad y \quad \widehat{\Delta u} = -|k|^2 \hat{u}(k)$$

$$\Rightarrow \hat{u}_t(k) + |k|^2 \hat{u}(k) = 0 \quad \forall \; k \in \mathbb{R}^d, \; t>0$$

$$\Rightarrow \hat{u} = C \exp(-|k|^2 t) \quad \text{donde}$$

$$C = \hat{u}(k,0) = \hat{f}(k) \quad \forall \; k \in \mathbb{R}^d$$

$$\Rightarrow \hat{u}(t,k) = \hat{f}(k) \exp(-|k|^2 t).$$

$$\Rightarrow F^*(\hat{u}) = f * F^*(\exp(-|k|^2 t))$$

Tomando $\alpha = \frac{1}{4t}$ en P2) (a),

$$u(t,x) = f * \left(\left(\frac{2}{4t}\right)^{d/2} \exp\left(-\frac{|\cdot|^2}{4t}\right)\right)(x)$$

$$= \frac{1}{(2t)^{d/2}} \int_{\mathbb{R}^d} f(y) \exp(-|x-y|^2/4t) \, dy.$$

Igual que antes, como $f \in S(\mathbb{R}^d) \Rightarrow u(\cdot, t) \in S(\mathbb{R}^d)$ y $u(x, \cdot) \in C^1$ para $x \in \mathbb{R}^d$ y por lo tanto, se justifican los pasos de antes.

Notar que si $t < 0 \Rightarrow \exp(-|k|^2 t) \notin S(\mathbb{R}^d)$

$\Rightarrow$ no puedo tomar antitransformada.

P4 Sea $f \in S(\mathbb{R}^d)$ y consideramos

$$\begin{cases} u_t - \Delta u = f & \text{en } \mathbb{R}^d \times (0, \infty) \\ u(x, 0) = 0, & x \in \mathbb{R}^d \end{cases}$$

Encontrar solución.

Por Duhamel: Sea $v(x, t, s)$ solución de

$$\begin{cases} u_t - \Delta u = 0 & \text{en } \mathbb{R}^d \times (s, \infty) \\ u(x, s) = f(x, s) & x \in \mathbb{R}^d \end{cases}$$

se cumple que

$$u(x,t) = \int_0^t v(x, t, s)\, ds$$

satisface $u_t - \Delta u = f$:

En efecto $u_t(x,t) = v(x,t,t) + \int_0^t v_t(x,t,s)\, ds$

$$= f(t) + \int_0^t \Delta v(x,t,s)\, ds$$

$$= f(x,t) + \Delta \int_0^t v(x,t,s)\, ds$$

$$= f(x,t) + \Delta u(x,t).$$

Como $\Phi(t,x) = \frac{1}{(2t)^{d/2}} \int_{\mathbb{R}^d} f(y) \exp(-|x-y|^2/4t)\, dy$

resuelve calor, proponemos

$$u(t,x) = \int_0^\infty \frac{1}{(2t)^{d/2}} \int_{\mathbb{R}^d} f(y) \exp(-|x-y|^2/4(t-s))\, dy$$

$\Rightarrow$ se cumple que $u_t - \Delta u = 0$. Resta ver que $u(x,0) = 0$.

Haciendo cambio de variable,

$$u(t,x) = \int_0^t \frac{1}{(2t)^{d/2}} \int_{\mathbb{R}^d} f(x-y) \exp(-|y|^2/4(t-s))\, dy$$

$$\Rightarrow |u(t,x)| = \int_0^t \frac{1}{(2t)^{d/2}} \int_{\mathbb{R}^d} |f(x-y)|\, |\exp(-|y|^2/4(t-s))|\, dy$$

$$\leq \|f\|_\infty \int_0^\infty \frac{1}{(2t)^{d/2}} \int_{\mathbb{R}^d} |\exp(-|y|^2/4(t-s))|\, dy$$

$$\frac{1}{(2t)^{d/2}}\int_{\mathbb{R}^d} e^{-|x|^2/4(t-s)}\,dx \underset{z=\frac{x}{2\sqrt{t}}}{\overset{=}{\downarrow}} \frac{2^d\,\cancel{t^{d/2}}}{(2\cancel{t})^{d/2}}\int_{\mathbb{R}^d} e^{-|z|^2}\,dz$$

$$\Rightarrow dz = \frac{1}{2\sqrt{t}^{\,d}}\,dx$$

$$= 2^{d/2}\prod_{i=1}^{d}\int_{\mathbb{R}} e^{-z_i^2}\,dz_i = 2^{d/2}\left(\sqrt{\pi}\right)^d$$

$$\Rightarrow |u(t,x)| \le \|f\|_\infty \int_0^t 2^{d/2}\left(\sqrt{\pi}\right)^d dt$$

$$= C\,T \xrightarrow[t\to 0]{} 0.$$