AUXILIAR 6

P1

(a). Por auxiliares anteriores, sabemos que

$$u(t,x) = \frac{1}{(2t)^{d/2}} \int_{\mathbb{R}^d} f(y) \exp(|x-y|^2/4t)\, dy$$

es solución y, por construcción, es única.

Veamos que $u \in C^\infty([0,\infty), S(\mathbb{R}^d))$.

Tenemos que $\hat{u}(t,\xi) = \hat{f}(\xi)\exp(-|\xi|^2 t)$. Luego, para $t,s \in [0,\infty)$,

$$\frac{1}{|h|} |\hat{u}(t+h, k) - \hat{u}(t,k)|_{\alpha,\beta}$$

$$= \frac{1}{|h|} \sup_{k\in\mathbb{R}^d} |k|^\alpha \left| \partial^\beta \left(e^{-|k|^2(t+h)} - e^{-|k|^2 t}\right) \hat{f}(k) \right|$$

$$\leq \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \sup_{k\in\mathbb{R}^d} |k|^\alpha \left| \frac{\partial^\gamma\left(e^{-|k|^2(t+h)} - e^{-|k|^2 t}\right)}{h} \right| |\partial^{\beta-\gamma}\hat{f}(k)|$$

$$\leq \underbrace{\sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \sup_{k\in\mathbb{R}^d}}\, |k|^\alpha \underbrace{\left| \left(\frac{\partial^\gamma\left(e^{-|k|(t+h)}\right) - \partial^\gamma\left(e^{-|k|^2 t}\right)}{h}\right) \right|}_{\longrightarrow 0} \underbrace{|\partial^{\beta-\gamma}\hat{f}(k)|}_{\text{acotado}}$$

①

$\Rightarrow u \in C([0,\infty), S(\mathbb{R}^d))$

Como $\partial_t u = \Delta u \in C([0,\infty), S(\mathbb{R}^d))$, se tiene que $u \in C^1([0,\infty), S(\mathbb{R}^d))$. Iterando, se tiene $u \in C^\infty([0,\infty), S)$.

(b). Sea $t \in [0,\infty)$, notemos que $u(t,\cdot) \in H^s(\mathbb{R}^d)$.

En efecto,

$$\|u(t)\|^2_{H^s(\mathbb{R}^d)} = \|e^{-t|k|^2} \langle k \rangle^s \hat{f}(k)\|^2_{L^2(\mathbb{R}^d)}$$

$$\leq C \|\langle k \rangle^s \hat{f}(k)\|^2_{L^2(\mathbb{R}^d)} = C \|f\|^2_{H^s(\mathbb{R}^d)}.$$

$e^{-t|k|^2} \in S(\mathbb{R}^d) \Rightarrow e^{-t|k|^2} \in L^\infty(\mathbb{R}^d)$

Sea $j > 0$,

$$\|\partial_t^j u(t)\|^2_{H^{s-2j}(\mathbb{R}^d)} = \|\Delta^j u(t)\|^2_{H^{s-2j}(\mathbb{R}^d)}$$

$$= \|\langle k \rangle^{s-2j} \widehat{\Delta^j u(t)}\|^2_{L^2(\mathbb{R}^d)}$$

$$\leq \|\langle k \rangle^{s-2j} \langle k \rangle^{2j} \hat{u}(t)\|^2_{L^2(\mathbb{R}^d)} = \|u\|^2_{H^s(\mathbb{R}^d)}$$

$$\leq C \|f\|^2_{H^s(\mathbb{R}^d)}$$

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P2|

Sean u_1, u_2 dos soluciones de (2). Definimos $w = u_1 - u_2$.

$\Rightarrow$ w es solución de (*) $\begin{cases} w_t - \Delta w + \mu w = 0 \\ w(0,x) = 0 . \end{cases}$

Definimos la energía como $E(t) = \int_{\mathbb{R}^d} |w|^2 dt$

Veamos que la energía es decreciente:

Multiplicamos (*) por w e integramos en x

$$0 = \int_{\mathbb{R}^d} w_t w \, dx - \int_{\mathbb{R}^d} \Delta w \, w \, dx + \mu \int_{\mathbb{R}^d} w w$$

$$= \frac{1}{2} \int_{\mathbb{R}^d} (|w|^2)_t \, dx + \int_{\mathbb{R}^d} |\nabla w|^2 dx + \mu \int_{\mathbb{R}^d} |w|^2 dx .$$

$$\Rightarrow \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |w|^2 dx = - \int_{\mathbb{R}^d} |\nabla w|^2 dx - \mu \int_{\mathbb{R}^d} |w|^2 dx$$

$$\leq 0, \quad \forall t \in [0, \infty).$$

$\Rightarrow E(t) \leq E(0)$.

$\Rightarrow \int_{\mathbb{R}^d} |w|^2 dx \leq \int_{\mathbb{R}^d} |w(0,x)|^2 dx = 0. \rightarrow w \equiv 0.$

P4|

Sea $x_0 \in \mathbb{R}^d$, $t_0 > 0$.

Definimos $$E(t) = \frac{1}{2} \int_{B(x_0, t_0 - t)} u_t^2(t,x) + |\nabla u(t,x)|^2 dx$$

para $t \in [0, t_0]$.

Luego,

$$E'(t) = \int_{B(x_0, t_0 - t)} u_t u_{tt} + \nabla u \cdot \nabla u_t \, dx - \frac{1}{2} \int_{\partial B(x_0, t_0 - t)} u_t^2 + |\nabla u|^2 ds$$

$$= \int_{B(x_0, t_0 - t)} u_t (\underbrace{\cancel{u_{tt} - \Delta u}}_{0}) dx - \frac{1}{2} \int_{\partial B(x_0, t_0 - t)} u_t^2 + |\nabla u|^2 ds$$

$$+ \int_{\partial B(x_0, t_0 - t)} \frac{\partial u}{\partial \nu} u_t \, ds$$

multiplicando (4) por u_t e IPP en x

$$\int_B u_t u_{tt} - \int_B u_t \Delta u = \int_B u_t u_{tt} + \int_B \nabla u_t \cdot \nabla u - \int_{\partial B} \frac{\partial u}{\partial \nu} u_t \, ds.$$

Tenemos que $\left| \underbrace{\frac{\partial u}{\partial \nu}}_{= \nabla u \cdot N} u_t \right| \leq |\nabla u| \, |u_t| \underset{(\text{Young})}{\leq} \frac{1}{2} (u_t^2 + |\nabla u|^2)$

$= \nabla u \cdot N$ donde N es normal de B de módulo 1.

luego

$$E'(t) = \int_{\partial B(x_0, t_0 - t)} \frac{\partial u}{\partial \nu} u_t - \frac{1}{2} u_t^2 - \frac{1}{2} |\nabla u|^2 \, dS$$

$$\leq \int_{\partial B(x_0, t_0 - t)} \frac{1}{2}\left(u_t^2 + |\nabla u|^2\right) - \frac{1}{2}\left(u_t^2 + |\nabla u|^2\right) dS$$

$$= 0 \quad \forall\, t \in [0, t_0]$$

$$\Rightarrow E(t) \leq E(0) \quad \forall\, t \in [0, t_0]$$

Como $E(0) = \frac{1}{2} \int_{B(x_0, t_0)} u_t^2 + |\nabla u|^2 \, dt = 0$

$\Rightarrow u_t \equiv 0$ y $\nabla u \equiv 0 \Rightarrow u \equiv 0$ en C.

PS1

(a) Basta probar $\operatorname{Im} \int u \bar{u}_{x_j} \, dx = 0$

Multiplicamos por $\bar{u}_{x_j}$, integramos y tomamos parte real

$$0 = \operatorname{Re} \int_{\mathbb{R}^d} i u_t u_{x_j} dx + \int_{\mathbb{R}^d} \Delta u \bar{u}_{x_j} \, dx + \int_{\mathbb{R}^d} |u|^{p-1} u \bar{u}_{x_j} \, dx$$

$= I_1 + I_2 + I_3$

Analizamos I_3:

$$\left(|u|^{p+1}\right)_{x_j} = \mathrm{Re}\left(\left(u\cdot\bar u\right)^{\frac{p+1}{2}}\right)_{x_j} = \mathrm{Re}\,\frac{p+1}{2}\left(u\,\bar u\right)^{\frac{p-1}{2}}\left(u\,\bar u\right)_{x_j}$$

$$= \mathrm{Re}\,\frac{p+1}{2}\left(u\bar u\right)^{\frac{p-1}{2}}\left(u_{x_j}\bar u + u\,\bar u_{x_j}\right)$$

$$= \mathrm{Re}\,(p+1)\left(u\bar u\right)^{\frac{p-1}{2}} u\,\bar u_{x_j} = \mathrm{Re}\,(p+1)\,|u|^{p-1}\,u\,\bar u_{x_j}$$

$$\Rightarrow \mathrm{Re}\,|u|^{p-1} u\,\bar u_{x_j} = \left(|u|^{p+1}\right)_{x_j}\frac{1}{p+1}$$

$$\Rightarrow \mathrm{Re}\int_{\mathbb{R}^d} |u|^{p-1}\,u\,\bar u_{x_j}\,dx = \frac{1}{p+1}\int_{\mathbb{R}^d}\left(|u|^{p+1}\right)_{x_j}\,dx$$

$$\underset{\substack{(\mathrm{IPP}\\ \text{sobre } 1)}}{=} 0$$

Analizamos I_2:

$$\mathrm{Re}\int_{\mathbb{R}^d} \Delta u\,\bar u_{x_j}\,dx = \sum_{i=1}^{d}\mathrm{Re}\int_{\mathbb{R}^d} u_{x_i x_i}\,\bar u_{x_j}\,dx$$

$$\underset{(\mathrm{IPP})}{=} -\sum_{i=1}^{d}\mathrm{Re}\int_{\mathbb{R}^d} u_{x_i}\left(\bar u_{x_i}\right)_{x_j}\,dx = -\frac{1}{2}\sum_{i=1}^{d}\int_{\mathbb{R}^d}\left(|u_{x_i}|^2\right)_{x_j}\,dx$$

$$\mathrm{Re}\left(v\,\bar v_{x_j}\right) = \frac{1}{2}\left(|v|^2\right)_{x_j}$$

$$\underset{(\mathrm{IPP})}{=} 0.$$

Analizamos I_1:

$$\operatorname{Im}\int_{\mathbb{R}^d} u_t \bar{u}_{x_j}\, dx = \operatorname{Im}\int_{\mathbb{R}^d} (u\bar{u}_{x_j})_t - \operatorname{Im}\int_{\mathbb{R}^d} u\bar{u}_{x_j,t}$$

$$(u\bar{u}_{x_j})_t = u_t \bar{u}_{x_j} + u\bar{u}_{x_j,t}$$

Falta ver que $\operatorname{Im}\int_{\mathbb{R}^d} u\bar{u}_{x_j t} = 0$

Notemos que, por (5),

$$u_{t x_j} = i\Delta u_{x_j} + i(|u|^{p-1}u)_{x_j}$$

$$\Rightarrow -\operatorname{Im}\int_{\mathbb{R}^d} u\bar{u}_{x_j t} = -\operatorname{Im}\int_{\mathbb{R}^d} \overline{i\Delta u_{x_j}}\, u - \operatorname{Im}\int_{\mathbb{R}^d} u\, \overline{i(|u|^{p-1}u)_{x_j}}$$

$$= \operatorname{Re}\int_{\mathbb{R}^d} \overline{\Delta u_{x_j}}\, u + \operatorname{Re}\int_{\mathbb{R}^d} u\, \overline{(|u|^{p-1}u)_{x_j}}$$

$$= \operatorname{Re}\int_{\mathbb{R}^d} \Delta u_{x_j}\, \bar{u} + \operatorname{Re}\int_{\mathbb{R}^d} \bar{u}\,(|u|^{p-1}u)_{x_j}$$

$= 0$ (analizando como en I_2 e I_3)

Por lo tanto, $0 = \operatorname{Im}\int_{\mathbb{R}^d} (u\bar{u}_{x_j})_t = \frac{d}{dt}\operatorname{Im}\int_{\mathbb{R}^d} u\bar{u}_{x_j}$

(b). Multiplicamos (3) por $\bar{u}_t$, integramos, tomamos parte real

$$0 = \mathrm{Re}\int_{\mathbb{R}^d} i u_t \bar{u}_t \, dx - \mathrm{Re}\int_{\mathbb{R}^d} \Delta u \, \bar{u}_t \, dx + \mathrm{Re}\int_{\mathbb{R}^d} |u|^{p-1} u \bar{u}_t \, dx$$

$$\underset{(IPP)}{=} \cancel{\mathrm{Re}\int_{\mathbb{R}^d} i |u_t|^2 \, dx}^{\,0} + \mathrm{Re}\int_{\mathbb{R}^d} \nabla u \, \nabla \bar{u}_t \, dx + \mathrm{Re}\int_{\mathbb{R}^d} |u|^{p-1} u \bar{u}_t \, dx$$

$$= +\frac{1}{2}\int_{\mathbb{R}^d} \left(|\nabla u|^2\right)_t + \frac{1}{p+1}\int_{\mathbb{R}^d} \left(|u|^{p+1}\right)_t \, dx$$

$$= \frac{d}{dt}\left(\int_{\mathbb{R}^d} \frac{1}{2} |\nabla u|^2 + \frac{|u|^{p+1}}{p+1} \, dx \right) = 0.$$