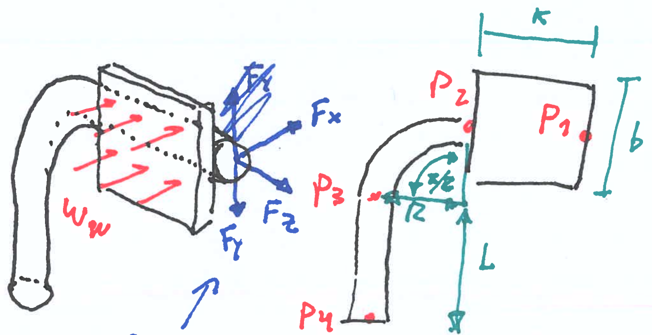
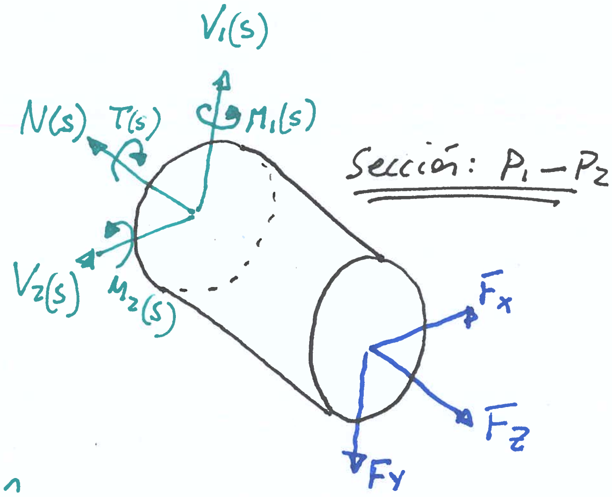
Auxiliar
Castigliano
Prob. 2.




Esfuerzos Ficticios para poder usar Castigliano.

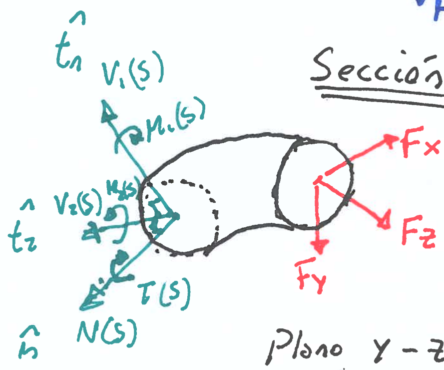



$0 < s < k$ | (sección $P_1 - P_2$)

$N(s) = F_z$ . $\quad M_1(s) = -\cancel{s}F_x - \dfrac{Wbs^2}{2}$

$V_1(s) = F_y \quad M_2(s) = sF_y$

$V_2(s) = F_x + Wbs \quad T(s) = 0$

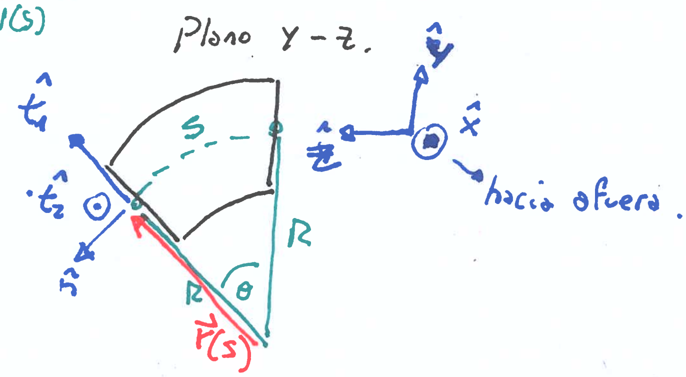



$0 < s < \frac{\pi R}{2}$ | (Sección $P_2 - P_3$)

$\boxed{s = R \cdot \theta}$

El vector que describe el ~~arc~~ arco que sigue la geometría.

$$\vec{r}(s) = R\cos\left(\frac{s}{R}\right)\hat{y} + R\sin\left(\frac{s}{R}\right)\hat{z}$$




Recordamos que:

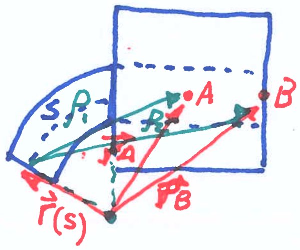
$$\hat{n} = \frac{d\vec{r}}{ds} \Big/ \left\|\frac{d\vec{r}}{ds}\right\|$$ es el vector normal a su trayectoria, entonces:

$$\hat{n} = \frac{-\sin\left(\frac{s}{R}\right)\hat{y} + \cos\left(\frac{s}{R}\right)\hat{z}}{\sqrt{\sin^2\left(\frac{s}{R}\right) + \cos^2\left(\frac{s}{R}\right)}} = -\sin\left(\frac{s}{R}\right)\hat{y} + \cos\left(\frac{s}{R}\right)\hat{z}$$

Ahora nos gustaría que $V_2$ continuara alineado al de la sección anterior luego $\hat{t}_2 = \hat{x}$

Entonces $$\hat{t}_1 = \hat{n} \times \hat{t}_2 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & -\sin & \cos \\ 1 & 0 & 0 \end{vmatrix} \begin{matrix} \hat{x} & \hat{y} \\ 0 & -\sin \\ 1 & 0 \end{matrix} = \begin{pmatrix} 0 \\ \cos(s/R) \\ \sin(s/R) \end{pmatrix}$$

Calculamos el radio $\vec{\rho}(s)$ desde el corte hasta las fuerzas.




$$\cancel{\vec{r}_A : \begin{pmatrix} 0 \\ R \\ K/2 \end{pmatrix}} \qquad \underbrace{\vec{r}_B = \begin{pmatrix} 0 \\ R \\ -K \end{pmatrix}} \qquad \underbrace{\vec{r}_A : \begin{pmatrix} 0 \\ R \\ -K/2 \end{pmatrix}}$$

Vector distancia centro circunferencia y Acción $\vec{F}$ ficticias

Vector distancia centro circunferencia y Acción Viento.

$$\vec{\rho}_A(s) = \vec{r}_A(s) - \vec{r}(s) = \begin{pmatrix} 0 \\ R \\ -K/2 \end{pmatrix} - \begin{pmatrix} 0 \\ R\cos\left(\frac{s}{R}\right) \\ R\sin\left(\frac{s}{R}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ R\left(1-\cos\left(\frac{s}{R}\right)\right) \\ -\left(\frac{K}{2}+R\sin\left(\frac{s}{R}\right)\right) \end{pmatrix}$$

$$\vec{\rho}_B(s) = \vec{r}_B(s) - \vec{r}(s) = \begin{pmatrix} 0 \\ R \\ -K \end{pmatrix} - \begin{pmatrix} 0 \\ R\cos\left(\frac{s}{R}\right) \\ R\sin\left(\frac{s}{R}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ R\left(1-\cos\left(\frac{s}{R}\right)\right) \\ -\left(K+R\sin\left(\frac{s}{R}\right)\right) \end{pmatrix}$$

Ahora:

* $N(s)$ y $T(s)$ viven en $\hat{n}$.
* $V_1(s)$ y $M_1(s)$ viven en $\hat{t}_1$
* $V_2(s)$ y $M_2(s)$ viven en $\hat{t}_2$

$$\begin{pmatrix} -F_x \\ -F_y \\ -F_z \end{pmatrix} \qquad \begin{pmatrix} -W_w K \cdot b \\ 0 \\ 0 \end{pmatrix}$$

Entonces.

$$\sum \vec{F} = 0 \ : \ N(s)\hat{n} + V_1(s)\cdot\hat{t}_1 + V_2(s)\cdot\hat{t}_2 + \vec{F}_{Fict.} + \vec{F}_W = 0.$$

Aprovechamos que $\hat{n} \perp \hat{t}_1 \perp \hat{t}_2$, entonces.

$$N(s) = -\left(\vec{F}_{Fict} + \vec{F}_W\right)\cdot\hat{n}$$
$$V_1(s) = -\left(\vec{F}_{Fict} + \vec{F}_W\right)\cdot\hat{t}_1$$
$$V_2(s) = -\left(\vec{F}_{Fict} + \vec{F}_W\right)\cdot\hat{t}_2$$

$$N(s) = F_z\cos\left(\frac{s}{R}\right) - F_y\sin\left(\frac{s}{R}\right)$$
$$V_1(s) = F_z\sin\left(\frac{s}{R}\right) + F_y\cos\left(\frac{s}{R}\right)$$
$$V_s(s) = F_x + W_w bK.$$

Realizando por otro lado la sumatoria de torques se obtiene:

$$T(s)\cdot\hat{n} + M_1(s)\,\hat{t}_1 + M_2(s)\,\hat{t}_2 + \vec{\rho}_1(s)\times\vec{F}_W + \vec{\rho}_2(s)\times\vec{F}_{Fict} = 0$$

$$\vec{P}_1(s) \times \vec{F}_W = \begin{pmatrix} 0 \\ R(1-\cos(\frac{s}{R})) \\ -(\frac{K}{2}+R\sin(\frac{s}{R})) \end{pmatrix} \times \begin{pmatrix} -W_w bK \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{K}{2}+R\sin(\frac{s}{R}) \\ R(1-\cos(\frac{s}{R})) \end{pmatrix}$$

$$\vec{P}_2(s) \times \vec{F}_{FICT} = \begin{pmatrix} 0 \\ R(1-\cos(\frac{s}{R})) \\ -(K+R\sin(\frac{s}{R})) \end{pmatrix} \times \begin{pmatrix} -F_x \\ -F_Y \\ -F_Z \end{pmatrix} = \begin{pmatrix} R(\cos(\frac{s}{R})-1)F_Z - (K+R\sin(\frac{s}{R}))F_Y \\ (K+R\sin(\frac{s}{R}))\cdot \\ R(1-\cos(\frac{s}{R}))F_x \end{pmatrix}$$

Al igual que el caso anterior se tendrá:

$$T(s) = \left(\vec{P}_1(s)\times\vec{F}_W - \vec{P}_2\times\vec{F}_{FICT}\right)\hat{n}.$$

$$M_1(s) = -\left(\vec{P}_1(s)\times\vec{F}_w + P_2(s)\times\vec{F}_W\right)\hat{t}_1$$

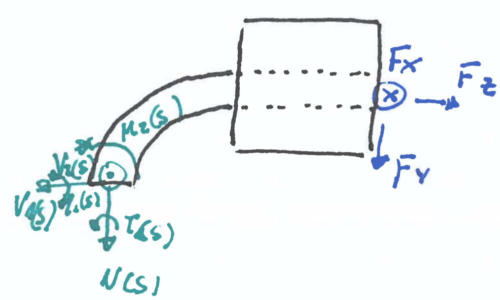
$$M_2(s) = -\left(\vec{P}_1(s)\times\vec{F}_W + \vec{P}_2(s)\times\vec{F}_{FICT}\right)\cdot\hat{t}_2$$

$$T(s) = (F_x + W_K b)R + K\sin\left(\frac{s}{R}\right)\left(F_x + \frac{W_K b}{2}\right) + R\cos\left(\frac{s}{R}\right)\left(F_x + W_w\right)$$

$$M_1(s) = -F_x\left(K\cos\left(\frac{s}{R}\right) + R\sin\left(\frac{s}{R}\right)\right) - W_w K b\left(\frac{K}{2}\cos\left(\frac{s}{R}\right) + R\sin\left(\frac{s}{r}\right)\right)$$

$$M_2(s) = -R\left(\cos\left(\frac{s}{R}\right) - 1\right)F_Z + \left(K+R\sin\left(\frac{s}{R}\right)\right)F_Y$$

$0 < s < L$ (sección $P_3 - P_4$)




$N(s) = -F_Y$

$V_1(s) = F_Z$

$V_2(s) = F_x$

$T(s) = (R+K)F_x + \left(R + \frac{K}{2}\right)W_{w}$

$M_1(s) = -(R+s)(F_x + W_N K b)$

$M_2(s) = RF_Z + (K+R)F_Y$

## ENERGÍA ACUMULADA Y CASTIGLIANO

$$\delta_i = \frac{\partial U}{\partial F_i} = \underbrace{\frac{1}{EA}\int_\Gamma N\frac{\partial N}{\partial F_i}ds}_{\text{Axial}} + \underbrace{\frac{1}{EI_z}\int_\Gamma M\frac{\partial M}{\partial F_i}ds}_{\text{Flexión}} + \underbrace{\frac{1}{GJ}\int_\Gamma T\frac{\partial T}{\partial F_i}ds}_{\text{Torsión}} + \underbrace{\frac{40}{3\pi D^2 G}\int_\Gamma V\frac{\partial V}{\partial s}ds}_{\text{Corte}}.$$

## DESPLAZAMIENTO EN X

Axial: $\partial_{F_x} U_N = \dfrac{1}{EA}\left[\int_0^K N \cancel{\dfrac{\partial N}{\partial F_x}}^{0} ds + \int_0^{\pi R/2} N \cancel{\dfrac{\partial N}{\partial F_x}}^{0} ds + \int_0^L N \cancel{\dfrac{\partial N}{\partial F_x}}^{0} ds\right] \boxed{= 0}$

Flexión: $\partial_{F_x} U_F = \dfrac{1}{EI_z}\left[\int_0^K M_1 \dfrac{\partial M_1}{\partial F_x} + M_2 \cancel{\dfrac{\partial M_2}{\partial F_x}}_{0} ds + \int_0^{\pi R/2} M_1 \dfrac{\partial M_1}{\partial F_x} + M_2 \cancel{\dfrac{\partial M_2}{\partial F_x}}_{0} ds + \int_0^L M_1 \dfrac{\partial M_1}{\partial F_x} + M_2 \cancel{\dfrac{\partial M_2}{\partial F_x}}_{0} ds\right]$

$= \dfrac{1}{EI_z}\left[\int_0^K \dfrac{wb}{2} s^3 + \cancel{F_x}^{0} s^2 ds + \int_0^{\pi R/2} \left\{\cancel{F_x}^{0}\left(K\cos\left(\tfrac{s}{R}\right) + R\sin\left(\tfrac{s}{R}\right)\right) + wbK\left(\tfrac{K}{2}\cos\left(\tfrac{s}{R}\right) + R\sin\left(\tfrac{s}{R}\right)\right)\right\} \cdot\right.$

$\left.\left(K\cos\left(\tfrac{s}{R}\right) + R\sin\left(\tfrac{s}{R}\right)\right) ds + \int_0^L (R+s)\cancel{F_x} + \left(R+\tfrac{K}{2}\right) w\, p \cdot b \cdot ds\right]$

$$\boxed{= \frac{1}{EI_z}\left[\frac{wbK^4}{8} + wK\cdot b\left(\frac{KR}{2} + 6 + \pi K^2 + 2\pi r^2\right)\cdot R + w_{Kb}\left(R^2L + RL^2 + \frac{L^3}{3}\right)\right]}$$

Torsión: $\dfrac{1}{GJ}\left[\int_0^K T \cancel{\dfrac{dT}{dF_x}}^{0} ds + \int_0^{\pi R/2} \left\{(\cancel{F_x}^{0} + wKB)R + K\sin\left(\tfrac{s}{R}\right)\left(\cancel{F_x}^{0} + \tfrac{wb}{2}\right) + R\cos(s/R)(\cancel{F_x}^{0} + Kw)\right\}(1)\right.$

$\left. + \int_0^L \left((R+K)(\cancel{F_x}^{0}) + \left(R+\tfrac{K}{2}\right)w_{AB}\right)(R+K)\, ds\right]$

$$\boxed{= \frac{1}{GJ}\left[wb\left(KR^2 + \pi K^2 R + 2\pi R^3\right) + \left(R+\frac{K}{2}\right)(R+K)wKbL\right]}$$

Corte: $\dfrac{40}{3\pi D^2 G}\left[\int_0^K V_1 \cancel{\dfrac{dV_1}{dF_x}}^{0} + V_2 \dfrac{dV_2}{dF_x} ds + \int_0^{\pi R/2} V_1 \cancel{\dfrac{dV_1}{dF_x}}^{0} + V_2 \dfrac{dV_2}{dF_x} ds + \int_0^L V_1 \cancel{\dfrac{\partial V_1}{\partial F_x}}^{0} + V_2 \dfrac{\partial V_2}{\partial F_x} ds\right]$

$$\boxed{= \frac{40}{3\pi D^2 G}\left[\int_0^K \cancel{F_x}^{0} + wb\, s\; ds + \int_0^{\pi R/2} \cancel{F_x}^{0} + wbK\, ds + \int_0^L \cancel{F_x}^{0}\, ds\right] = \frac{40}{3\pi D^2 G}\left[\frac{K^2}{2} + \frac{K\pi R}{2}\right]}$$

# DESPLAZAMIENTO Y

Axial: $\partial_{F_Y} U_N = \frac{1}{EA}\left[\int_0^K N \cancel{\frac{\partial N}{\partial F_Y}} ds + \int_0^{\pi R/2} N \frac{\partial N}{\partial F_Y} ds + \int_0^L N \frac{\partial N}{\partial F_Y} ds\right]$

$= \frac{1}{EA}\left[\int_0^{\pi R/2}\left(\cancel{F_z}\cos\left(\frac{s}{R}\right) - \cancel{F_Y}\sin\left(\frac{s}{R}\right)\right)\cos\left(\frac{s}{R}\right) ds + \int_0^L \cancel{F_Y}\, ds\right] = \boxed{0}$ .

Flexión: $\partial_{F_Y} U_F = \frac{1}{EI_z}\left[\int_0^K M_1 \cancel{\frac{dM_1}{\partial F_Y}} + M_2 \frac{\partial M_2}{\partial F_Y} ds + \int_0^{\pi R/2} M_1 \cancel{\frac{\partial M_1}{\partial F_Y}} + M_2 \frac{\partial M_2}{\partial F_Y} ds + \int_0^L M_1 \cancel{\frac{\partial M_1}{\partial F_Y}} + M_2 \frac{\partial M_2}{\partial F_Y} ds\right]$

$= \frac{1}{EI_z}\left[\int_0^K \cancel{F_Y}\cdot s\, ds + \int_0^{\pi R/2}\left\{\cancel{F_Y}\left(K + R\sin\left(\frac{s}{R}\right)\right) - \cancel{F_z}R\left(\cos\left(\frac{s}{R}\right) - 1\right)\right\}\left(K + R\sin\left(\frac{s}{R}\right)\right) ds + \int_0^L \left(R\cancel{F_z} + (K+R)\cancel{F_Y}\right)(K+R)\, ds\right]$

$\boxed{\Rightarrow \partial_{F_Y} U_F = 0}$

Torsión: $\partial_{F_Y} U_T = \frac{1}{GJ}\left[\int_0^K T \cancel{\frac{\partial T}{\partial F_Y}} ds + \int_0^{\pi R/2} T \cancel{\frac{\partial T}{\partial F_Y}} ds + \int_0^L T \cancel{\frac{\partial T}{\partial F_Y}} ds\right] = \boxed{0}$

Corte: $\partial_{F_Y} U_C = \frac{40}{3\pi D^2 G}\left[\int_0^K V_1 \frac{\partial V_1}{\partial F_Y} + V_2 \cancel{\frac{\partial V_2}{\partial F_Y}} ds + \int_0^{\pi R/2} V_1 \frac{\partial V_1}{\partial F_Y} + V_2 \cancel{\frac{\partial V_2}{\partial F_Y}} ds + \int_0^L V_1 \cancel{\frac{\partial V_1}{\partial F_Y}} + V_2 \cancel{\frac{\partial V_2}{\partial F_Y}} ds\right]$

$= \frac{40}{3\pi D^2 G}\left[\int_0^K \cancel{F_Y}\, ds + \int_0^{\pi R/2}\left(\cancel{F_z}\sin\left(\frac{s}{R}\right) + \cancel{F_Y}\cos\left(\frac{s}{R}\right)\right)\cos\left(\frac{s}{R}\right) ds\right] = \boxed{0}$

$\Rightarrow \delta_Y = 0$ .

Con el mismo procedimiento se obtiene $\delta_z = 0$ .