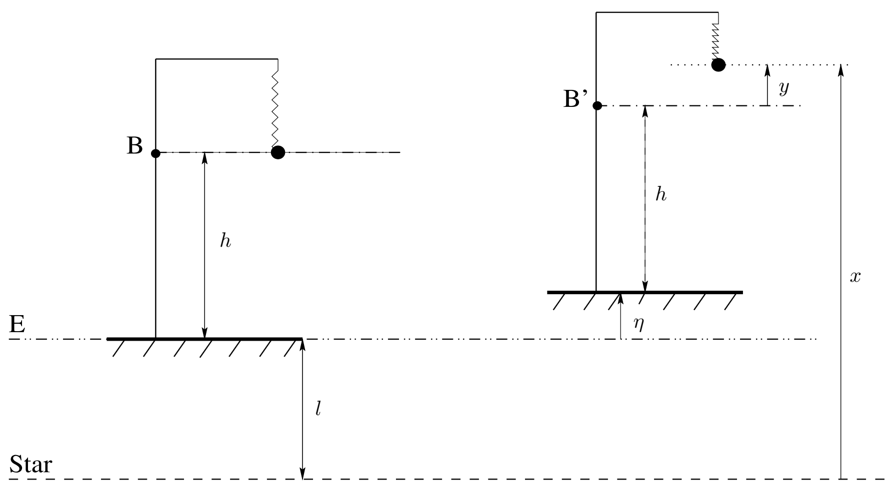
Problem 2.2: (French 4-6)[1] Seismograph

a) The displacement of mass M relative to the earth is y and η is the displacement of the earth's surface relative to the distant stars. Let x be the distance of mass M relative to the distant stars.



Left Figure: The horizontal dashed line through E is the equilibrium position of the earth relative to the star. The horizontal dashed line through B is the equilibrium position of the mass relative to the star. It is also the equilibrium position of the mass relative to the seismometer.

Right Figure: The dashed line through E, is the same as in the left figure. B is now a distance η farther away from the star than B (we indicate this with B'). The dashed line through B' is no longer the equilibrium position of the mass relative to the star, but it is the equilibrium position relative to the seismometer.

We can see from the figures that: $x = l + y + h + \eta$ or $\ddot{x} = \ddot{\eta} + \ddot{y}$. Newton's 2nd law only applies to an inertial reference frame. The acceleration of M is $\ddot{x}$. However, the spring force and the damping force depend on the displacement and velocity relative to the Earth (i.e. relative to B'). The amount by which the length of the spring changes is y in both reference frames (that of the star and that of the seismograph). Thus the magnitude of the spring force is ky. Since it is assumed that the air inside the closed box of the seismograph follows the motion of the Earth, the

[1]The notation "French" indicates where this problem is located in one of the textbooks used for 8.03 in 2004: French, A. P. *Vibrations and Waves. The M.I.T. Introductory Physics Series.* Cambridge, MA: Massachusetts Institute of Technology, 1971. ISBN-10: 0393099369; ISBN-13: 9780393099362.

damping force is $-b\dot{y}$. Notice, if the air does not follow the Earth then the damping force would be $-b(\dot{y}+\dot{\eta})$. Hence: $M\ddot{x} = -ky - b\dot{y} \quad 0 = \ddot{\eta} + \ddot{y} + \frac{k}{M}y + \frac{b}{M}\dot{y}$

$-\ddot{\eta} = \ddot{y} + \gamma\dot{y} + \omega_0^2 \quad$ or $\quad \frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \omega_0^2 y = -\frac{d^2\eta}{dt^2}$ where $\gamma = \frac{b}{m}$ and $\omega_0^2 = \frac{k}{m}$.

b) Steady state solution for y when $\eta = C\cos(\omega t)$.

$$\eta \quad = \quad Ccos(\omega t) \qquad \frac{d^2\eta}{dt^2} = -C\omega^2\cos(\omega t) \qquad \frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \omega_0^2 y = C\omega^2\cos(\omega t) \tag{2}$$

To solve the equation using the complex exponential method we reframe the above equation as follows $\frac{d^2z}{dt^2} + \gamma\frac{dz}{dt} + \omega_0^2 z = C\omega^2 e^{i\omega t}$. Let $z = Ae^{i(\omega t-\delta)}$ be the solution to the above equation. Now $y = Re(z)$. Substituting these in Eq. 2.

$$\begin{aligned}(-\omega^2 A + i\gamma\omega A + \omega_0^2 A)e^{i(\omega t-\delta)} &= C\omega^2 e^{i\omega t} \\ (\omega_0^2 - \omega^2)A + i\gamma\omega A &= C\omega^2 e^{i\delta}\end{aligned}$$

Equating the real and imaginary parts of the equation we get:

$$(\omega_0^2 - \omega^2)A \quad = \quad C\omega^2\cos\delta \qquad \gamma\omega A = C\omega^2\sin\delta$$

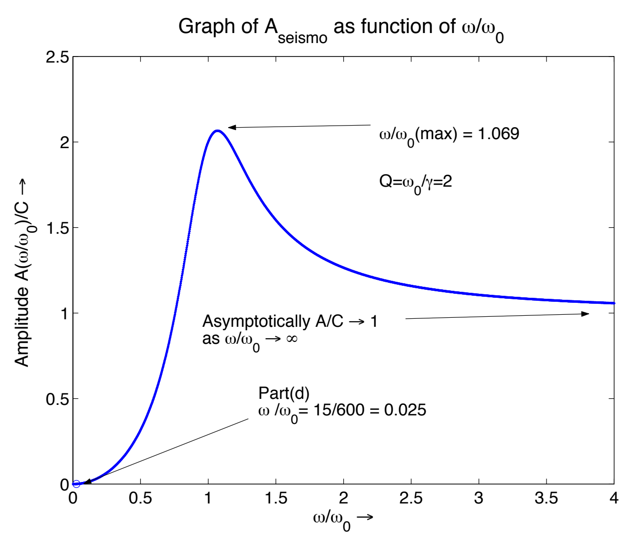
Therefore the steady state solution for y is $y = A\cos(\omega t - \delta)$ where

$$A(\omega) \quad = \quad \frac{C\omega^2}{[(\omega_0^2-\omega^2)^2 + (\gamma\omega)^2]^{\frac{1}{2}}} \qquad \tan\ \delta(\omega) = \frac{\gamma\omega}{\omega_0^2 - \omega^2}$$

Behavior of $A(\omega)$ for various values of ω

$$\omega \to 0 \quad A \to 0 \qquad \omega \to \omega_0 \quad A \to QC \qquad \omega \to \infty \quad A \to C$$

c) The graph of the amplitude A of the displacement y (in units of C) as a function ω is shown to the right. Note: Q=ω_0/γ is taken to be 2.



d) Period of the Seismograph T_s is 30 s and Q is 2.

$$T_s \quad = \quad 2\pi/\omega_0 = 30 \text{ s} \qquad \omega_0 = \frac{2\pi}{30} = \frac{\pi}{15} \text{ rad/s} \qquad \gamma = \frac{\omega_0}{Q} = \frac{\pi/15}{2} = \frac{\pi}{30} \text{ rad/s}$$

Now the time period of oscillations of the earth's surface is 20 min and the amplitude of maximum acceleration is 10^{-9} m/s^2.

$$\omega = \frac{2\pi}{T_s} = \frac{2\pi}{1200} = \frac{\pi}{600} \text{ rad/s} \qquad a_{max} = C\omega^2 = 10^{-9} \text{ m/s}^2 \qquad C = \frac{a_{max}}{\omega^2} = 3.6 \times 10^{-5} \text{ m}$$

Substituting values for ω, ω_0, γ and C in the equation for amplitude A we get:

$$A(\omega) = \frac{C\omega^2}{[(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2]^{\frac{1}{2}}} \qquad A = 2.28 \times 10^{-8} \text{ m}$$

Notice that C (amplitude of the Earth's oscillations) is about 1600 times larger than A. It seems to us that this is a very poorly designed seismometer. Values of A of the order of 2.3×10^{-8} m must be observable for this tremor to be detected. If the frequency of the oscillations $\omega \gg \omega_0$ the value of $A \to C$ (see the figure for Part (c) above). The amplitude of the earthquake oscillations can then directly be read off the seismometer.