TEORIA DE CRANDAL-RABINOWITZ

$S(\lambda, u) = 0$, $S : X \to Y$

$S(\lambda, 0) = 0, \quad \forall \lambda$

$DS(\lambda^*, 0) = L$ NO ES INVERTIBLE

(1') $\dim N_L = 1$, $N_L = \{ t u^*, t \in \mathbb{R} \}$

(2') $\exists \psi \in Y^*$, $\psi \neq 0$

$R_L = \operatorname{Ran} L = \{ y \in Y \mid \langle y, \psi \rangle = 0 \}$

LYAPUNOV - SCHMIDT:

$Q S(\lambda, u) = 0$

Q : PROYECCIÓN A $\operatorname{Ran} L \subseteq Y$

$Q S(\lambda, v + w(\lambda, v)) = 0$

$v \in \operatorname{Ker} L = N_L$

EN TORNO DE λ^*

$w(\lambda^*, 0) = 0$

$D_v w(\lambda^*, 0) = 0$

$P S(\lambda, v + w(\lambda, v)) = 0$

P: PROYECCIÓN A N_L

$\Longleftrightarrow$

$\beta(\lambda, t) = \langle \psi, S(\lambda, t u^* + w(\lambda, t u^*)) \rangle = 0$

$$v = t u^*, \quad t \in \mathbb{R}$$

$$\beta(\lambda, 0) = 0$$

$$\partial_t \beta(\lambda, t) = \langle \psi, D_u S(\lambda, tu^* + w(\lambda, tu^*)) h^*(t) \rangle$$

$$h^*(t) = u^* + D_v w(\lambda, tu^*) u^*$$

$$\partial_t \beta(\lambda^*, 0) = \langle \psi, D_u S(\lambda^*, 0) u^* \rangle$$

$$= \langle \psi, L u^* \rangle$$

$$L u^* \in \operatorname{Ran} L \Rightarrow \langle \psi, L u^* \rangle = 0$$

$$\partial_t \beta(\lambda^*, 0) = 0$$

$$\partial_t \beta(\lambda, 0) = \langle \psi, D_u S(\lambda, \overbrace{w(\lambda, 0)}^{0}) h^*(0) \rangle$$

$$= \langle \psi, D_u S(\lambda, 0)[u^* + D_v w(\lambda, 0) u^*] \rangle$$

$$\partial_{\lambda t} \beta(\lambda, 0) = \partial_\lambda \langle \psi, D_u S(\lambda, 0)[u^* + D_v w(\lambda, 0) u^*]$$

$$= \langle \psi, D_{u\lambda} S(\lambda, 0)[u^* + D_v w(\lambda, 0) u^*] \rangle$$

$$+ \langle \psi, D_u S(\lambda, 0) D_{\lambda v} w(\lambda, 0) u^* \rangle$$

$\lambda = \lambda^*$

$$\partial_{\lambda t} \beta(\lambda^*, 0) = \langle \psi, D_{u\lambda} S(\lambda^*, 0) u^* \rangle$$

$$+ \langle \psi, \underbrace{\underbrace{D_u S(\lambda^*, 0)}_{L} D_{\lambda v} w(\lambda^*, 0) u^*}_{\in \operatorname{Ran} L} \rangle$$

$\Longrightarrow$

$$\partial_{\lambda t} \beta(\lambda^*, 0) = \langle \psi, D_{u\lambda} S(\lambda^*, 0) u^* \rangle$$

TEOREMA

SI $\quad D_{u\lambda} S(\lambda^*, 0) u^* \notin \operatorname{Ran} L$

$\Longrightarrow$ λ^* ES UN PUNTO DE BIFURCACIÓN.

DEMOSTRACIÓN

RESOLVER $\beta(\lambda, t)$ LOCALMENTE CERCA $\lambda = \lambda^*$, $t = 0$.

(¿ $\lambda = \lambda(t)$?)

FORMAL: $\beta(\lambda, t) = t\, h(\lambda, t)$

BASTA RESOLVER $h(\lambda, t) = 0$, $\lambda = \lambda(t)$

$$\partial_\lambda h(\lambda,0)\Big|_{\lambda=\lambda^*} \neq 0$$

DEFINIMOS:

$$h(\lambda,t) = \begin{cases} \dfrac{\beta(\lambda,t)}{t} & t\neq 0 \\ \partial_t \beta(\lambda,0) & , t=0 \end{cases}$$

h ES CLASE C^1 EN TORNO DE $(\lambda^*,0)$.

POR EJEMPLO: $t\neq 0$

$$\partial_t h(\lambda,t) = \frac{\partial_t \beta(\lambda,t)}{t} - \frac{\beta(\lambda,t)}{t^2}$$

$\Longrightarrow$

$\lambda = \lambda^* \Longrightarrow$

$$\partial_t h(\lambda^*,t) = \frac{\partial_t \beta(\lambda^*,t)}{t} - \frac{\beta(\lambda^*,t)}{t^2}$$

$$\beta(\lambda^*,0) = 0 = \partial_t \beta(\lambda^*,0)$$

$$\partial_t h(\lambda,t) = \frac{\partial_t\beta(\lambda,t)}{t} - \frac{\partial_t \beta(\lambda,t)t + o(t)t}{t^2}$$

$$= \frac{o(t)}{t} \xrightarrow[t\to 0]{} 0$$

$$\partial_\lambda h(\lambda,t) = \frac{\partial_\lambda \beta(\lambda,t)}{t} \xrightarrow[\substack{t\to 0\\ \lambda\to\lambda^*}]{} \partial_{\lambda t}\beta(\lambda^*,0)$$

$$\partial_\lambda h(\lambda^*,0) = \partial_{\lambda t}\beta(\lambda^*,0)$$

$$= \langle \psi, D_{u\lambda} S(\lambda^*,0)u^* \rangle \neq 0$$

(HIPÓTESIS)

$\Longrightarrow$ $\exists$ $\varepsilon > 0$ $\lambda : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$

$\lambda = \lambda(t)$ TAL QUE:

$\lambda(0) = \lambda^*$

$h(\lambda(t), t) = 0$

$\Longrightarrow \beta(\lambda(t), t) = 0 \Longrightarrow$

$u(t) = tu^* + w(\lambda(t) + tu^*)$

ES UNA SOLUCIÓN DE LA ECUACIÓN DE BIFURCACIÓN.

$\Longrightarrow$

$S(\lambda(t), tu^* + w(\lambda(t) + tu^*)) = 0$

$t \in (-\varepsilon, \varepsilon)$

$$(\lambda(t), u(t)) \xrightarrow[t\to 0]{} (\lambda^*, 0)$$

$u(t) \neq 0, \quad t \neq 0$

COMENTARIOS SOBRE LA CURVA

$$\Sigma_S = (t u^*, \lambda(t)) \in X_L \times \mathbb{R}$$

(i) Σ_S ES SUAVE.

(ii) COMPORTAMIENTO DE $\lambda(t)$ CUANDO $t \to 0$: $\lambda(t) =: \lambda^* + \rho(t)$

$$\lambda(t) = \lambda^* + \lambda'(0)t + o(t)$$

FUNCIÓN IMPLICITA:

$$\lambda'(0) = -\frac{\partial_t h(\lambda^*, 0)}{\partial_\lambda h(\lambda^*, 0)}$$

$$\partial_\lambda h(\lambda^*, 0) = \langle \psi, \underbrace{D_{u\lambda} S(\lambda^*, 0)}_{a} u^* \rangle \neq 0$$

$\boxed{\text{CONDICIÓN DE TRANSVERSALIDAD}}$

$$\partial_t h(\lambda^*,0) = \frac{1}{2}\partial^2_{tt}\beta(\lambda^*,0)$$

$$= \lim_{t\to 0} \frac{\frac{\beta(\lambda^*,t)}{t} - \partial_t\beta(\lambda^*,0)}{t}$$

$$\beta(\lambda,t) = \beta(\lambda,0) + \partial_t\beta(\lambda,0)t + \frac{1}{2}\partial^2_{tt}\beta(\lambda,0)t^2 + o(t^2)$$

ENTONCES:

$$\partial_t\beta(\lambda,t) = \langle\psi, D_u S(\lambda, tu^* + w(\lambda,tu^*))\phi(t)\rangle$$

$$\phi(t) = u^* + D_v w(\lambda,tu^*)u^*$$

$$\partial_t\phi(t) = D_{vv} w(\lambda,tu^*)[u^*,u^*]$$

$$= D_{vv} w(\lambda,tu^*)[u^*]^2$$

$$\partial_{tt}\beta(\lambda,t) = \langle\psi, D_{uu} S(\lambda, tu^* + w(\lambda,tu^*))[\phi(t)]^2\rangle$$

$$+ \langle\psi, D_u S(\lambda, tu^* + w(\lambda,tu^*))\overbrace{[u^*]^2}^{D_{vv}w(\lambda,tu^*)}\rangle$$

$\lambda = \lambda^*$

$\Longrightarrow$

$$\partial_{tt}\beta(\lambda^*,0) = \langle \psi, D_{uu} S(\lambda^*,0)[u^*]^2 \rangle$$

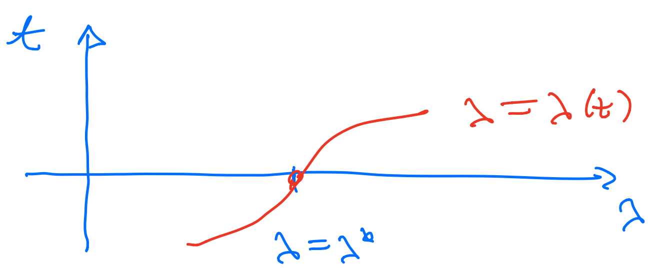
$$+ \langle \psi, \underbrace{L\, D_{vv} w(\lambda^*,0)[u^*]^2}_{\text{Ran } L} \rangle$$

$$\partial_{tt}\beta(\lambda^*,0) = \langle \psi, D_{uu} S(\lambda^*,0)[u^*]^2 \rangle$$

$$b = \frac{1}{2}\partial_{tt}\beta(\lambda^*,0)$$

Si $b \neq 0 \implies$

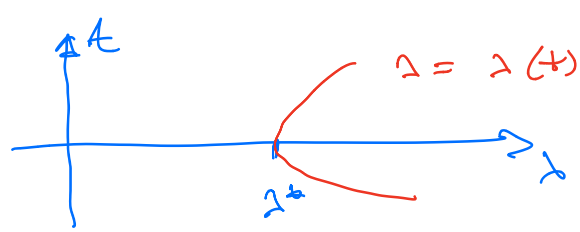
$$\lambda(t) = \lambda^* - \frac{b}{a} t + o(t^2), \quad t \to 0$$



BIFURCACIÓN SE DICE <u>TRANSCRÍTICA</u>

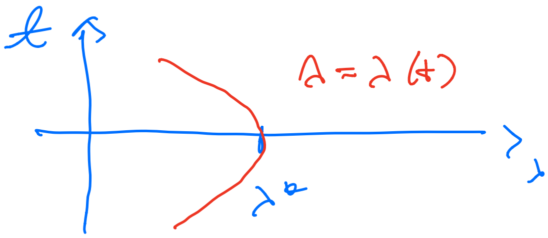
EN MUCHOS CASOS $b = 0$

$$\implies \lambda(t) = \lambda^* + \frac{1}{2}\lambda''(0) t^2 + \ldots$$



Si $\lambda''(0) = c > 0$

SUPERCRÍTICA



SI $\lambda''(0) < 0$

SUBCRITICA

TEOREMA

SEA $T \in C^2(X, Y)$, $T(0) = 0$

$D_u T(0) = 0$

ENTONCES TODO VALOR CARACTERISTICO SIMPLE λ^0 DE A ES UN PUNTO DE BIFORCACIÓN DE

$$S(\lambda, u) = 0$$

$$S(\lambda, u) = u - \lambda A u - T(u)$$

BASTA VERIFICAR LA CONDICIÓN DE TRANSVERSALIDAD:

$$D_{u\lambda} S(\lambda^0, 0) u^0 \notin R_L \quad , \quad L = D_u S(\lambda^0, 0)$$

TENEMOS:

$$D_u S(\lambda, 0) h = h - \lambda A h$$

$$D_{\lambda u} S(\lambda, 0) u^0 = -A u^0$$

PERO λ^* ES UN VALOR CARACTERISTICO

$$\Rightarrow \lambda^* A u^* = u^* \in N_L, \quad L = I - \lambda^* A$$

$$\Rightarrow A u^* \in N_L \Rightarrow A u^* \notin \operatorname{Ran} L = R_L$$

$$L u^* = 0 \Rightarrow u^* - \lambda^* A u^* = 0$$

$$A u^* = \frac{1}{\lambda^*} u^* \in N_L$$