No-Lineal
Lunes 30 nov.

Aplicación rearrangements

Desigualdad de Nash. (Lieb-Loss 8.13)

Sea $f \in H^1 \cap L^2$

entonces $\|f\|_{L^2}^{1+2/d} \leq C_n \|\nabla f\|_{L^2} \|f\|_{L^1}^{2/d}$

Sea $f \in H^1(\mathbb{R}^d, \mathbb{C})$, entonces $|f| \in H^1(\mathbb{R}^d)$ y

$$\|\nabla |f|\|_{L^2} \leq \|\nabla f\|_{L^2}$$

Demo

$$g_\varepsilon = \sqrt{|f|^2+\varepsilon^2} - \varepsilon$$

$$g_\varepsilon^2 = |f|^2+\varepsilon^2+\varepsilon^2 - 2\varepsilon\sqrt{|f|^2+\varepsilon^2}$$

$$= |f|^2 + 2\varepsilon\left(\varepsilon - \sqrt{|f|^2+\varepsilon^2}\right) \leq |f|^2$$

$$\nabla g_\varepsilon = \frac{\bar{f}\nabla f + f\nabla \bar{f}}{2\sqrt{|f|^2+\varepsilon^2}} = \frac{1}{\sqrt{|f|^2+\varepsilon^2}} \mathrm{Re}(\bar{f}\nabla f)$$

$$|\nabla g_\varepsilon|^2 = \frac{1}{|f|^2+\varepsilon^2} \mathrm{Re}(\bar{f}\nabla f)^2 = \frac{1}{|f|^2+\varepsilon^2}\left\{\mathrm{Re}(f)\,\mathrm{Re}(\nabla f) + \mathrm{Im}(f)\,\mathrm{Im}(\nabla f)\right\}^2$$

$$\left\{\mathrm{Re}^2 f\ \mathrm{Re}^2(\nabla f) + \mathrm{Im}^2 f\ \mathrm{Im}^2 \nabla f + 2\,\mathrm{Im}(f)\,\mathrm{Re} f\ \mathrm{Im} \nabla f\ \mathrm{Re}\nabla f\right.$$

Por ver:

$$\{\quad\}^2 \leq |f|^2 |\nabla f|^2 = \left[(\mathrm{Re} f)^2 + (\mathrm{Im} f)^2\right]\left[(\mathrm{Re}\nabla f)^2 + (\mathrm{Im}\nabla f)^2\right]$$

$$\{\ \}^2 \leq (\mathrm{Re} f)^2 \mathrm{Re}(\nabla f)^2 + (\mathrm{Im} f)^2 (\mathrm{Im} \nabla f)^2 + \mathrm{Im}(\nabla f)^2 (\mathrm{Re} f)^2 + \mathrm{Im}(f)^2 \mathrm{Re}(\nabla f)^2 \}$$

$$= |f|^2 |\nabla f|^2 .$$

$$|\nabla g_\varepsilon|^2 \leq \frac{|f|^2 |\nabla f|^2}{|f|^2 + \varepsilon^2} \leq |\nabla f|^2$$

$$\int |\nabla g_\varepsilon|^2 \leq \int |\nabla f|^2$$

$$\int |g_\varepsilon|^2 \leq \int f^2$$

$g_\varepsilon \to |f|$ puntualmente

↳ conv. dominada. $\square$

Paso 2. Sea $f \in H^1 \cap L^2$. $f \neq 0$.

$$\underbrace{\|\nabla f\|_{L^2} \cdot \|f\|_{L^1}^{2/d}}_{\|f\|_{L^2}^{1+2/d}} \geq \underbrace{\|\nabla f^*\|_{L^2}}_{\|f^*\|_{L^2}^{1+2/d}} \cdot \|f^*\|_{L^2}$$

Podemos restringirnos a f no-negativa, con simetría esférica y decreciente radialmente.

Sea $R>0$.

$$f = \underbrace{f\mathbb{1}_{B(0,R)}}_{g} + \underbrace{f\mathbb{1}_{B^c(0,R)}}_{h} \qquad \bar g = \frac{1}{|B(0,R)|}\int_{B(0,R)} g$$

$$\int f^2 = \int g^2 + \int h^2$$

$$= \int (g-\bar g+\bar g)^2 + \int h^2$$

$$= \int (g-\bar g)^2 + 2\bar g \cancel{\int (g-\bar g)} + \bar g^2 |B(0,R)|^2 + \int h^2$$

$h \le \bar g$ por radialmente decreciente.

$$\le \int (g-\bar g)^2 + \bar g^2 |B(0,R)| + \bar g \int h$$

Ingrediente 3

$$\int (g-\bar g)^2 \le \frac{1}{\mu_N(B(0,R))}\int |\nabla g|^2 = \frac{R^2}{\underbrace{\mu_N(B(0,1))}_{\alpha}}\int |\nabla g|^2$$

$$\mu_N(\Omega) = \inf_{\substack{f\in H^1(\Omega)\\ \int f = 0}} \frac{\int |\nabla f|^2}{\int f^2}$$

$\downarrow$

1er v.p. de Neuman.

$$\Big(\underbrace{\int f}_{|B(0,R)|}\Big)^2 = \Big(\int g + \int h\Big)^2 = \big(\underbrace{|B(0,R)|\cdot \bar g}\big)^2 + \Big(\int h\Big)^2 + 2|B(0,R)|\,\bar g \int h$$

$$|B_R|\,\bar g^2 + \bar g \int h + \bar g \int h + \Big(\int h\Big)^2$$

$$\int f^2 \leq \frac{R^2}{\alpha} \int_{B(0,R)} |\nabla g|^2 + \frac{\left(\int f\right)^2}{|B(0,R)|}$$

$$\leq \frac{R^2}{\alpha} \int |\nabla f|^2 + \frac{\left(\int f\right)^2}{\omega_d R^d}$$

$$\ell(R) = C_1 R^2 + C_2 R^{-d}$$

$$\ell'(R) = 2 C_1 R - d C_2 R^{-d-1} \qquad R_* = \frac{d\, C_2}{2 C_1}$$

$$\ell(R_*) = C_1^{\frac{d}{d+2}} \, C_2^{\frac{2}{d+2}} \left(\left(\frac{d}{2}\right)^{\frac{2}{d+2}} + \left(\frac{2}{d}\right)^{\frac{d}{d+2}} \right)$$

$$\leq \frac{\|\nabla f\|_{L^2}^{2 \cdot \frac{d}{d+2}}}{\alpha^{\frac{d}{d+2}}} \cdot \frac{\|f\|_{L^1}^{2 \cdot \frac{2}{d+2}}}{\omega_d^{\frac{2}{d+2}}} \cdot \left(\quad \right)$$

$$\|f\|_{L^2}^2 \leq C_d \, \|\nabla f\|_{L^2}^{2 \frac{d}{d+2}} \, \|f\|_{L^1}^{2 \cdot \frac{2}{d+2}} \rightarrow \square$$

$$f = \left(\varphi_N(x) - \varphi_N(1)\right) \cdot \mathbb{1}_{B(0,1)}$$

primer f.p de Neuman en la bola. $B(0,1)$.

$$\mathcal{E}[\rho] = \underbrace{\int \rho^{5/3} + C_W \int |\nabla \sqrt{\rho}|^2}_{} - \int \frac{Z\rho}{|x|} + \frac{1}{2} D(\rho,\rho)$$

$$\frac{1}{2} D(\rho,\rho) = \frac{1}{2} \iint \frac{\rho(x)\rho(y)}{|x-y|}\,dx\,dy$$

princ. de exclusión de Pauli

Heisenberg

(ρ_n) suc. minim.

$(\sqrt{\rho_n})$ acotada en H^1 $\qquad \sqrt{\rho_n} \rightarrow u$ débil en H^1

$\Downarrow$ Fatou $\quad u^2 = \rho_0$

$$\mathcal{E}(\rho_0) \le \inf_{\rho \in \mathcal{D}_N} \mathcal{E}(\rho).$$

$$\tilde{E}_{Z,N} = \inf_{\rho \in \mathcal{B}_N} \mathcal{E}_Z(\rho)$$

$$E_{Z,N} = \inf_{\rho \in \mathcal{D}_N} \mathcal{E}_Z(\rho).$$

Prop

a) $\tilde{E}_{Z,N} = E_{Z,N} < 0$ si $Z, N > 0$.

b) $E_{Z,N}$ es convexo en N

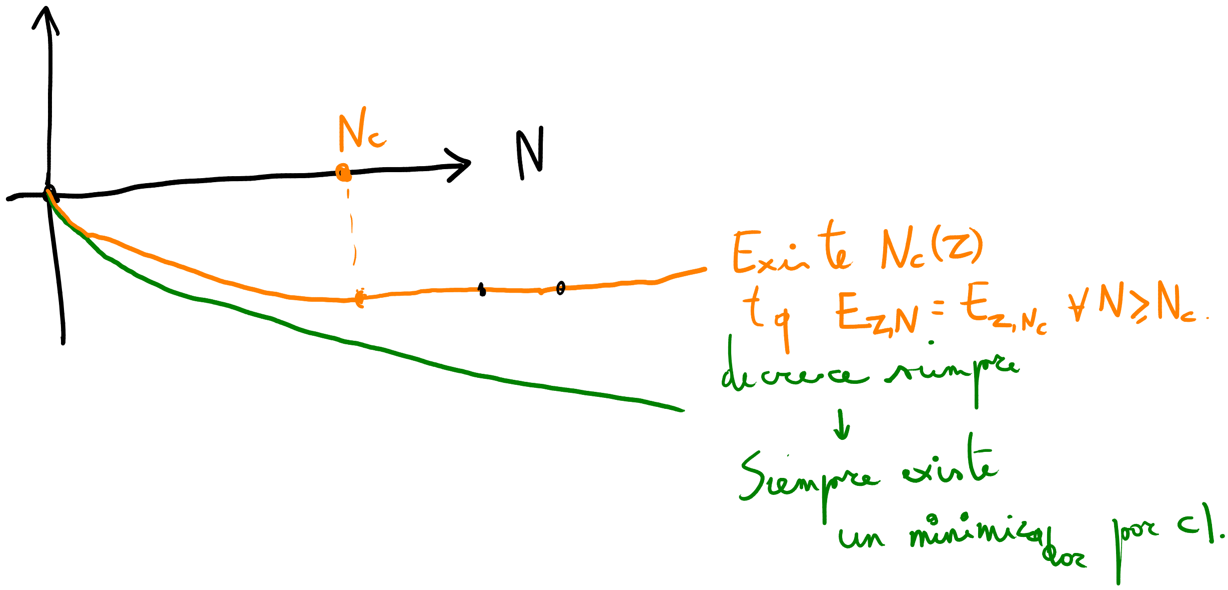
no-creciente en N

continua en N.

Strict binding inequality

c) Si $E_{Z,N} < E_{Z,N'}$ para $N' < N$, existe minimizador para $E_{Z,N}$.

$Z>0$ fijo

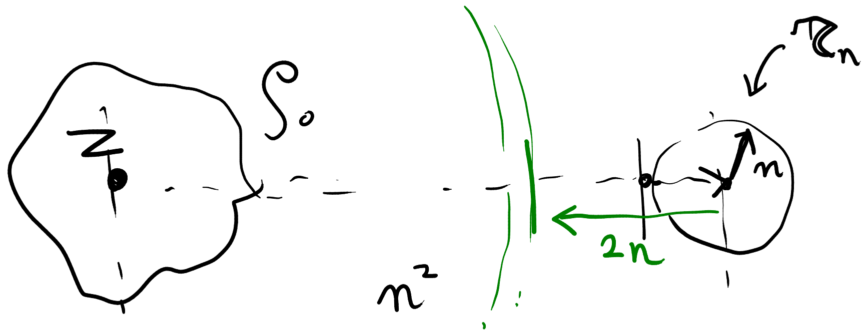


a) $\tilde{E}_{Z,N} \leq E_{Z,N}$ ya que $\mathcal{D}_N \subset \mathcal{B}_N$

Sea ρ_0 el minimizador para $\tilde{E}_{Z,N}$.

Si $\int \rho_0 = N$, $E_{Z,N} \leq \mathcal{E}_Z(\rho_0) = \tilde{E}_{Z,N}$

Si $\int \rho_0 = N - \delta$ con algún $\delta > 0$.



$\varphi \geq 0$, $\varphi \in \mathcal{D}(B(0,1))$, $\int \varphi = \delta$

$$\rho_n = \rho_0 + \frac{1}{n^3}\varphi\Big(\overbrace{\frac{\cdot - n^2 e_1}{n}}^{\tau_n}\Big)$$

$$\int \rho_n = \int \rho_0 + \frac{1}{n^3}\int \varphi\left(\frac{\cdot}{n}\right) = N$$

$$E_{Z,N} \leq \liminf_{n\to\infty} \mathcal{E}_Z(\rho_n)$$

$$\|\rho_n\|_{L^{5/3}} \leq \|\rho_n - \rho_0\|_{L^{5/3}} + \|\rho_0\|_{L^{5/3}}$$

$$\|\rho_n - \rho_0\|_{L^{5/3}}^{5/3} = \frac{1}{n^5}\int \varphi^{5/3}\left(\frac{x}{n}\right)dx$$

$$= \frac{1}{n^2}\|\varphi\|_{L^{5/3}}^{5/3}$$

$$\liminf_{n\to\infty} \|\rho_n\|_{L^{5/3}} \leq \|\rho_0\|_{L^{5/3}}.$$

$$\liminf_{n\to\infty} \int |\nabla\sqrt{\rho_0 + r_n}|^2 \leq \liminf_{n\to\infty} \int|\nabla\sqrt{\rho_0}|^2 + \int|\nabla\sqrt{r_n}|^2$$

$$\int|\nabla\sqrt{r_n}|^2 = \frac{1}{n^3}\cdot\frac{1}{n^2}\int|\nabla\sqrt{\varphi}|^2\left(\frac{x}{n}\right)dx$$

$$= \frac{1}{n^2}\int|\nabla\sqrt{\varphi}|^2$$

$$\int\frac{\rho_n}{|x|} = \int\frac{\rho_0}{|x|} + \int\frac{r_n(x)}{|x|} \geq \int\frac{\rho_0}{|x|}$$

$$D(\rho_n,\rho_n) = D(\rho_0,\rho_0) + \underbrace{D(r_n,r_n)}_{\downarrow\ \text{tiende a cero escalamiento}} + 2D(\rho_0,r_n)$$

$$2D(\rho_0,r_n)$$

Teorema de Newton

Si ρ tiene simetría esferica,

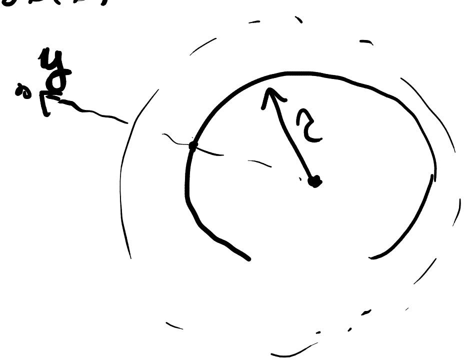
$$(|\cdot|^{-1} * \rho)(y) = \frac{1}{|y|} \int_{B(0,|y|)} \rho(x)\,dx + \int_{B(0,|y|)^c} \frac{\rho(x)}{|x|}\,dx$$

$$= \int \frac{\rho(x)\,dx}{\max(|x|,|y|)}$$

Aplicación de principio del máximo.

$$\int \frac{\rho(x)}{|x-y|}\,dx = \int_0^{+\infty} r^2 dr\, \rho(r) \int_{\partial B(0,1)} \frac{1}{|r\omega - y|}\,d\omega$$

Si $|y| > r$,



$\frac{1}{|x-y|}$ es analítica en $B(0,r)$

$$\Downarrow$$

el promedio sobre la esfera es igual al valor en el centro

Si $|y| < r$ $\quad \int \frac{1}{|x-y|}\,dy = \frac{1}{r}$

ρ_0 es radial (ya que es único.)

$$D(\rho_0, r_n) = \int r_n(y) \left(\int_{B(0,n^2-n)} \frac{\rho_0(x)}{|x-y|}\,dx \right) dy + \int r_n(y) \cdot \left(\int_{B(0,n^2-n)^c} \frac{\rho_0(x)}{|x-y|}\,dx \right) dy$$

$$\leq \int \frac{\tau_n(y)}{|y|} \int_{B(0,n^2-n)} \rho_0(x)\,dx \; dy$$

$$+ D(\tau_n, \rho_0 \mathbb{1}_{B^c(0,n^2-n)})$$

$$\leq \frac{N}{n^2-n} + \underbrace{\|\tau_n\|_{L^{\frac{6}{5}}}}_{\text{tiende a cero por cambio de var.}} \|\rho_0 \mathbb{1}_{B^c(0,n^2-n)}\|_{L^{\frac{6}{5}}}$$

a cero por conv. dominada

Hardy - Littlewood - Sobolev. $p, q > 1$.

$$\iint \frac{f(x)\,g(y)}{|x-y|^{\lambda}} \leq C \|f\|_{L^p} \|g\|_{L^q}$$

$$\frac{1}{p} + \frac{1}{q} + \frac{\lambda}{d} = 2$$

$$p = q = \frac{6}{5}$$

$$E_{Z,N} \leq \liminf_{n\to\infty} \mathcal{E}_Z[\rho_n] = \mathcal{E}_Z[\rho_0] = \tilde{E}_{Z,N}$$

$$\underline{E_{Z,N} = \tilde{E}_{Z,N}.}$$

b). $\tilde{E}_{Z,N}$ es decreciente en N. non-increasing

$N_1 < N_2$

$B_{N_1} \subset B_{N_2}$

$\tilde{E}_{Z,N_1} \geq \tilde{E}_{Z,N_2}$

convexo. $N_1, N_2 > 0$, $N = tN_1 + (1-t)N_2$

ρ_{N_1}, ρ_{N_2} min de $\tilde{E}_{Z,N_1}$ y $\tilde{E}_{Z,N_2}$. $\rho_t = t\rho_{N_1} + (1-t)\rho_{N_2}$

$$\tilde{E}_{Z,N} \leq \mathcal{E}_Z(\rho_t) \leq t\tilde{E}_{Z,N_1} + (1-t)\,\tilde{E}_{Z,N_2}$$

↑ no estricta ya que puede ser $\rho_{N_1} = \rho_{N_2}$.

<u>Continuidad</u> N_1, N_2, p.q. $N_1 < N_2$ $|N_1 - N_2| < \delta$

$$E_{Z,N_2} \leq E_{Z,N_1} \underset{?}{\leq} E_{Z,N_2} + \varepsilon$$

$\frac{N_1}{N_2}\rho_{N_2}$

$$E_{Z,N_1} \leq \mathcal{E}_Z\left[\frac{N_1}{N_2}\rho_{N_2}\right]$$

$$= \left(\frac{N_1}{N_2}\right)^{5/3}\int \rho_{N_2}^{5/3} + \frac{N_1}{N_2}\int|\nabla\sqrt{\rho_{N_2}}|^2 + \dots$$

$$\leq \mathcal{E}_Z[\rho_{N_2}] + \max\left[\left|\left(\frac{N_1}{N_2}\right)^{5/3} - 1\right|\int\rho_{N_2}^{5/3},\ \left|\frac{N_1}{N_2} - 1\right|\int|\nabla\sqrt{\rho_{N_2}}|^2, \dots\right]$$

$$\leq \mathcal{E}_Z[\rho_{N_2}] + \varepsilon$$

$\rho \to \alpha\rho$

$$\mathcal{E}_Z(\alpha\rho) = \alpha^{5/3}\int\rho^{5/3} + \alpha\left(\int|\nabla\sqrt{\rho}|^2 - Z\int\frac{\rho^2}{|x|}(x)\,dx\right)$$
$$+\alpha^2 D(\rho,\rho).$$

E_j^0 tomar $\varphi = \sqrt{\rho} = e^{-\gamma|x|}$, $\left(-\Delta - \frac{Z}{|x|}\right)\varphi = -\frac{Z^2}{4}\varphi$ (SE para hidrógeno)

$$E_{Z,N} \leq \mathcal{E}(\alpha\cdot\varphi^2) < 0.$$

c). Si no existe min. para $E_{Z,N}$,

entonces $\int \rho_N = N' < N$

$\uparrow$

min. para $\tilde{E}_{Z,N}$

Y entonces $E_{Z,N} = \mathcal{E}_Z(\rho_N) \geq E_{Z,N'}$

Contradice $E_{Z,N'} > E_{Z,N}$ si $N' > N$.