# Aux 5

**P1)** $X_i \sim U(\theta, \theta+1)$ $\rightsquigarrow$ $\max X_i - \min X_i \leq 1$

$$\Rightarrow f_\theta(x_i) = \mathbb{1}_{\{\theta \leq x_i \leq \theta+1\}}$$

$$L(\theta|x) = \prod_{i=1}^{n} f_\theta(x_i) = \prod_{i=1}^{n} \mathbb{1}_{\{\theta \leq x_i \leq \theta+1\}}$$

$$= \mathbb{1}_{\{\min x_i \geq \theta,\ \max x_i \leq \theta+1\}}$$

- De aquí, es evidente que $\max L(\theta|x) = 1$

- Si yo elijo $\theta = \min X_i$

$\Rightarrow L(\theta|x) = \mathbb{1}\{\underbrace{\min X_i \geq \theta}_{\checkmark}, \underbrace{\max X_i \leq \theta + 1}_{\checkmark}\} = 1$

- Si elijo $\theta = \max X_i - 1$

$\Rightarrow L(\theta|x) = \mathbb{1}\{\min X_i \geq \theta, \max X_i \leq \theta + 1\} = 1$

- En gral, cualquier $\theta \in [\max X_i - 1, \min X_i]$

$\Rightarrow L(\theta|x) = 1$

$\therefore \alpha = \max X_i - 1$

$\beta = \min X_i$

P2) $X_i \sim \text{Poisson}(\theta)$ $\quad X = (X_1, \dots, X_n)$ $\quad \theta \in \mathbb{R}_+$

$$\hookrightarrow f_\theta(x_i) = \frac{e^{-\theta}\,\theta^{x_i}}{x_i!}$$

$$\Rightarrow L(\theta|x) = \prod_{i=1}^{n} \frac{e^{-\theta}\,\theta^{x_i}}{x_i!} = \frac{e^{-n\theta}\,\theta^{\sum x_i}}{\prod x_i!}$$ Verosimilitud

$$\Rightarrow \ell(\theta|x) = \ln(L) = -n\theta + \sum x_i \ln(\theta) - \sum \ln(x_i!)$$ Log-verosimilitud

- Para buscar el óptimo en este caso basta derivar e igualar a 0.

- Sabemos que la función de densidad de la Poisson es cóncava, pues hereda las propiedades o "forma" de la dist. Normal, al ser su versión discreta//

$$\frac{\partial \ell}{\partial \theta} = -n + \frac{\sum x_i}{\theta} \overset{\text{Impongo}}{=} 0$$

$$\Rightarrow \theta_{EMV} = \frac{\sum x_i}{n} = \overline{x}$$

* Veamos que es máx:

$$\frac{\partial\left(\frac{\partial \ell}{\partial \theta}\right)}{\partial \theta} = -\frac{\sum x_i}{\theta^2} \leq 0$$

$\therefore$ $\ell$ es cóncava $\forall \theta$, y por lo tanto el pto.

$\theta_{EMV} = \bar{X}$ es un máximo $\blacksquare$

B) a) • Ya sabemos que los EMV de una gaussiana sin restricciones en los parámetros son

$$\boxed{\mu_1 = \frac{\sum x_i}{n}, \quad \sigma_1^2 = \frac{\sum (x_i - \bar{x})^2}{n}}$$

• Ahora, tenemos la restricción de que $\mu \geq \mu_0$

∴ En caso de que $\bar{X} < \mu_0$, la mejor estimación sería $\mu_0$

$$\Rightarrow \mu_{EMV} = \bar{X}\, \mathbb{1}_{\{\bar{X} \geq \mu_0\}} + \mu_0\, \mathbb{1}_{\{\bar{X} < \mu_0\}}$$

- Más formalmente, como estamos en un modelo Gaussiano:

$$f_{\mu,\sigma}(x_i) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}} \Longrightarrow L(\mu,\sigma|X) = \frac{1}{(2\pi)^{n/2}\sigma^n} e^{\frac{-1}{2\sigma^2}\sum (x_i-\mu)^2}$$

$$\Rightarrow \ell(\mu,\sigma|X) = \frac{-n}{2}\ln(2\pi) - n\ln(\sigma) - \frac{1}{2\sigma^2}\sum (x_i-\mu)^2$$

- Luego tenemos el problema

$$\begin{aligned} \min \quad & -\ell(\mu,\sigma|X) \\ \text{sa} \quad & \mu_0 - \mu \leq 0 \end{aligned}$$

$\Longrightarrow$ Por KKT:

1) $-\frac{\partial \ell}{\partial \mu}(\mu^*,\sigma|X) - \lambda = 0$

2) $\lambda(\mu_0 - \mu^*) = 0$

$\lambda = 0)$ Si $\lambda = 0 \Rightarrow \frac{\partial \ell}{\partial \mu}(\mu^*,\sigma|X) = 0 \Rightarrow \mu^* = \bar{X}$

$\lambda > 0$ ) Por (2) tenemos $\mu_0 - \mu^* = 0 \Rightarrow \mu^* = \mu_0$

- Notemos que el primer caso ocurre cuando $\mu^* > \mu_0$, es decir:

$$\mu^* > \mu_0 \Rightarrow \mu^* = \overline{X}$$

$\therefore$ $\boxed{\mu_{EMV} = \overline{X}\, \mathbb{1}_{\{\overline{X} \geq \mu_0\}} + \mu_0\, \mathbb{1}_{\{\overline{X} < \mu_0\}}}$ $\boxed{\mathrm{P}}$

P3) b) $\mathbb{E}(\hat{\mu}) = \mathbb{E}\left(\bar{X}\,\mathbb{1}_{\{\bar{X}\geq\mu_0\}} + \mu_0\,\mathbb{1}_{\{\bar{X}<\mu_0\}}\right)$

$$= \mathbb{E}\left(\bar{X}\,\mathbb{1}_{\{\bar{X}\geq\mu_0\}}\right) + \mu_0\,\mathbb{E}\left(\mathbb{1}_{\{\bar{X}<\mu_0\}}\right)$$

$$= \mathbb{E}\left(\bar{X}\,\mathbb{1}_{\{\bar{X}\geq\mu_0\}}\right) + \mu_0\,\mathbb{P}\left(\bar{X}<\mu_0\right)$$

- Recordemos $\bar{X}\sim N(\mu, \sigma^2/n) \Rightarrow Y := \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$

$$\therefore\ \mathbb{E}(\hat{\mu}) = \underbrace{\mathbb{E}\left(\bar{X}\,\mathbb{1}_{\{\bar{X}\geq\mu_0\}}\right)}_{(*)} + \mu_0\,\underbrace{\mathbb{P}\left(Y < \frac{\mu_0-\mu}{\sigma/\sqrt{n}}\right)}_{\text{Conocida}}$$

Conocida $= \Phi\left(\frac{\mu_0-\mu}{\sigma/\sqrt{n}}\right) = \int_{-\infty}^{\frac{\mu_0-\mu}{\sigma/\sqrt{n}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\,dy$

$$\otimes = \frac{\sigma}{\sqrt{n}} \mathbb{E}\left( \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \mathbb{1}\left\{ \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \geq \frac{\mu_0-\mu}{\sigma/\sqrt{n}} \right\} \right) + \mu \, \mathbb{E}\left( \mathbb{1}\left\{ \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \geq \frac{\mu_0-\mu}{\sigma/\sqrt{n}} \right\} \right) \qquad \bullet \; Y \sim N(0,1)$$

$$= \frac{\sigma}{\sqrt{n}} \mathbb{E}\left( Y - Y \mathbb{1}_{\left\{ Y < \frac{\mu_0-\mu}{\sigma/\sqrt{n}} \right\}} \right) + \mu \left[ 1 - \mathbb{P}\left( Y < \frac{\mu_0-\mu}{\sigma/\sqrt{n}} \right) \right]$$

$$= \sigma/\sqrt{n} \, \cancel{\mathbb{E}(Y)} - \sigma/\sqrt{n} \, \mathbb{E}\left( Y \mathbb{1}_{\left\{ Y < \frac{\mu_0-\mu}{\sigma/\sqrt{n}} \right\}} \right) + \mu - \mu \, \Phi\left( \frac{\mu_0-\mu}{\sigma/\sqrt{n}} \right)$$

$$= -\sigma/\sqrt{n} \int_{-\infty}^{\frac{\mu_0-\mu}{\sigma/\sqrt{n}}} y \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \, dy + \mu - \mu \, \Phi\left( \frac{\mu_0-\mu}{\sigma/\sqrt{n}} \right)$$

$$\therefore \; \boxed{\mathbb{E}(\hat{\mu}) = -\sigma/\sqrt{n} \int_{-\infty}^{\frac{\mu_0-\mu}{\sigma/\sqrt{n}}} y \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \, dy + (\mu_0 - \mu) \, \Phi\left( \frac{\mu_0-\mu}{\sigma/\sqrt{n}} \right) + \mu}$$

- Ahora veamos que $\mathbb{E}(\hat{\mu}) \longrightarrow \mu$ cuando $\mu_0 \longrightarrow -\infty$

$$\mathbb{E}(\hat{\mu}) = \underbrace{-\frac{\sigma}{\sqrt{n}} \int_{-\infty}^{\frac{\mu_0-\mu}{\sigma/\sqrt{n}}} y \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy}_{1} \underbrace{+ (\mu_0 - \mu)\, \Phi\left(\frac{\mu_0-\mu}{\sigma/\sqrt{n}}\right)}_{2} + \mu$$

(1) se va a 0 con $\mu_0 \longrightarrow -\infty$ pues solo afecta los límites de integración, no las cantidades integradas

$$\Rightarrow \int_{-\infty}^{\frac{\mu_0-\mu}{\sigma/\sqrt{n}}} y \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \xrightarrow{\mu_0 \to -\infty} 0$$

(2)

$$\lim_{\mu_0 \to -\infty} (\mu_0-\mu)\,\Phi\left(\frac{\mu_0-\mu}{\sigma/\sqrt{n}}\right) = \lim_{\mu_0 \to -\infty} \frac{\displaystyle\int_{-\infty}^{\frac{\mu_0-\mu}{\sigma/\sqrt{n}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\,dy}{1/(\mu_0-\mu)} \overset{L'H}{=} \lim_{\mu_0 \to -\infty} \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{\left(\frac{\mu_0-\mu}{\sigma/\sqrt{n}}\right)^2}{2}} \cdot \frac{\sqrt{n}}{\sigma}}{-1/(\mu_0-\mu)^2} = 0$$

$$\therefore\ \mathbb{E}(\hat{\mu}) \xrightarrow{\mu_0 \to -\infty} \mu \quad \blacksquare$$

P3) c) Queremos ver que $\mathbb{P}(\lim_n \hat{\mu} = \mu) = 1$

$$\circ\ \mathbb{P}\left(\lim_n \bar{X}\mathbb{1}_{\{\bar{X}\geq\mu_0\}} + \mu_0\mathbb{1}_{\{\bar{X}<\mu_0\}} = \mu\right)$$

$$= \mathbb{P}\left(\lim_n \bar{X}\mathbb{1}_{\{\bar{X}\geq\mu_0\}} + \mu_0\mathbb{1}_{\{\bar{X}<\mu_0\}} \,\middle|\, \lim_n \bar{X} \geq \mu_0\right)\mathbb{P}\left(\lim_n \bar{X} \geq \mu_0\right)$$

$$+ \mathbb{P}\left(\lim_n \bar{X}\mathbb{1}_{\{\bar{X}\geq\mu_0\}} + \mu_0\mathbb{1}_{\{\bar{x}<\mu_0\}} \,\middle|\, \lim_n \bar{X} < \mu_0\right)\mathbb{P}\left(\lim_n \bar{X} < \mu_0\right)$$

$$= \mathbb{P}\left(\lim_n \bar{X} = \mu \,\middle|\, \lim_n \bar{X} \geq \mu_0\right)\mathbb{P}\left(\lim_n \bar{X} \geq \mu_0\right) + \mathbb{P}\left(\lim_n \mu_0 = \mu \,\middle|\, \lim_n \bar{X} < \mu_0\right)\mathbb{P}\left(\lim_n \bar{X} < \mu_0\right)$$

$$= \underbrace{\mathbb{P}\left(\lim_n \bar{X} = \mu\right)}_{1}\underbrace{\mathbb{P}\left(\lim_n \bar{X} \geq \mu_0\right)}_{1} + \underbrace{\mathbb{P}\left(\mu_0 = \mu \,\middle|\, \lim_n \bar{X} < \mu_0\right)}_{0}\underbrace{\mathbb{P}\left(\lim_n \bar{X} < \mu_0\right)}_{0} = 1$$

LGN: 1    1    0    0

$\blacksquare$