$$0 \leq \mathbb{E}(Z^2)$$

$$= \mathbb{E}\left[\left(X - Y\frac{\mathbb{E}(XY)}{\mathbb{E}(Y^2}\right)^2\right]$$

Aux 6

P1) a) $\mathrm{Var}_\theta(r) = \mathbb{E}(r^2) - \mathbb{E}(r)^2$

- $\mathbb{E}_\theta(r(x,\theta,h)) = \int_{\mathbb{R}} \frac{f(x,\theta+h) - f(x,\theta)}{\cancel{f(x,\theta)}} \cancel{f(x,\theta)}\, dx$

$$= \int_{\mathbb{R}} f(x,\theta+h) - \int_{\mathbb{R}} f(x,\theta)$$

$$= 1 - 1 = 0$$

- $\mathrm{Var}_\theta(r) = \mathbb{E}_\theta(r^2) - \mathbb{E}_\theta(r)^2$

$$= I_{\theta,h} \;//$$

- $\mathrm{Cov}_\theta(r,\hat\theta) = \mathbb{E}_\theta\left[\left(r - \cancel{\mathbb{E}(r)}^{\to 0}\right)\left(\hat\theta - \cancel{\mathbb{E}(\hat\theta)}^{\to \theta}\right)\right]$

$$= \mathbb{E}_\theta\left[r(\hat\theta - \theta)\right]$$

$$= \mathbb{E}_\theta(r\hat\theta) - \mathbb{E}_\theta(r\theta)$$

$$= \underbrace{\mathbb{E}_\theta(r\hat\theta)}_{(1)} - \theta\, \cancel{\mathbb{E}_\theta(r)}^{\to 0}$$

$$\mathbb{E}_\theta(r\hat{\theta}) = \int_{\mathbb{R}} \left(\frac{f(x,\theta+h) - f(x,\theta)}{\cancel{f(x,\theta)}} \right) \hat{\theta}(x)\, \cancel{f(x,\theta)}\, dx$$

$$= \underbrace{\int_{\mathbb{R}} \hat{\theta}(x) f(x,\theta+h)\, dx}_{} - \underbrace{\int_{\mathbb{R}} \hat{\theta}(x) f(x,\theta)\, dx}_{}$$

$$= \mathbb{E}_{\theta+h}(\hat{\theta}) - \mathbb{E}_\theta(\hat{\theta})$$

$$= \theta + h - \theta = h \,//$$

$$\mathrm{Cov}(r,\hat{\theta})^2 \le \underbrace{\mathrm{Var}(r)}_{} \cdot \mathrm{Var}(\hat{\theta}) \qquad (*)$$

$$\parallel$$

$$h^2 \le I_{\theta,h} \cdot \mathrm{Var}(\hat{\theta}) \qquad \forall \theta, h$$

$$\therefore \quad \frac{h^2}{I_{\theta,h}} \le \mathrm{Var}(\hat{\theta}) \qquad \forall \theta, h$$

(*) Sean X, Y va. Def $Z = X - Y \dfrac{\mathbb{E}(XY)}{\mathbb{E}(Y^2)}$

$$
\begin{aligned}
\circ\ 0 &\le \mathbb{E}(Z^2) \\
&= \mathbb{E}\left[\left(X - Y \frac{\mathbb{E}(XY)}{\mathbb{E}(Y^2)}\right)^2\right] \\
&= \mathbb{E}(X^2) - 2\,\mathbb{E}(XY) \frac{\mathbb{E}(XY)}{\mathbb{E}(Y^2)} + \cancel{\mathbb{E}(Y^2)} \frac{\mathbb{E}(XY)^2}{\mathbb{E}(Y^2)^{\cancel{2}}} \\
&= \mathbb{E}(X^2) - \frac{\mathbb{E}(XY)^2}{\mathbb{E}(Y^2)}
\end{aligned}
$$

$$\therefore\ \mathbb{E}(XY)^2 \le \mathbb{E}(X^2)\,\mathbb{E}(Y^2) \quad \blacksquare$$

P1) b) En este caso $f(x,\vartheta) = \frac{1}{\vartheta} \mathbb{1}_{(0,\vartheta)}(x)$

$$\Rightarrow I_{\vartheta,h} \doteq \mathbb{E}(r^2) = \int_{\mathbb{R}} \left[\frac{\frac{1}{\vartheta+h} - \frac{1}{\vartheta}}{\frac{1}{\vartheta}} \right]^2 \frac{1}{\vartheta} \mathbb{1}_{(0,\vartheta)}(x)\, dx$$

$$= \left(\frac{\frac{1}{\vartheta+h} - \frac{1}{\vartheta}}{\frac{1}{\vartheta}} \right)^2 = \left(\frac{\vartheta}{\vartheta+h} - 1 \right)^2$$

$$= \left(\frac{\vartheta - (\vartheta+h)}{\vartheta+h} \right)^2 = \left(\frac{-h}{\vartheta+h} \right)^2 = \frac{h^2}{(\vartheta+h)^2}$$

P3) c) $\lim_{h \to 0^+} \frac{I_{\theta,h}}{h^2} = \lim_{h \to 0^+} \frac{h^2/(\theta+h)^2}{h^2} = \lim_{h \to 0^+} \frac{1}{(\theta+h)^2} = \frac{1}{\theta^2}$ ∎

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P2) a) $T(X) = \frac{2}{\theta} \sum X_i$

1) Es claro que $T(X)$ es estricto decreciente s/r a θ

2) Sabemos que $X_i \sim \exp(\theta)$

$$\Rightarrow \frac{2}{\theta} X_i \sim \exp\left(\frac{2}{\theta} \cdot \theta\right) \Rightarrow \frac{2}{\theta} X_i \sim \exp(2)$$

- Sabemos que suma de exponenciales es Gamma

$$\Rightarrow \sum \frac{2}{\theta} X_i = \frac{2}{\theta} \sum X_i = T(X) \sim \Gamma(n, 2)$$

$$\Rightarrow f_{\frac{2}{\theta}\sum X_i}(t) = \frac{1}{2^n \Gamma(n)} t^{n-1} e^{-t/2} \mathbb{1}_{t>0}(t)$$

↳ Esta ley no depende de θ

$\therefore T(X)$ es un pivote

↳ Tambien, de la def. del enunciado, es claro que es la densidad de una χ^2 con $2n$ grados de libertad

$$\therefore T(X) \sim \chi^2(2n)$$

P2) b) Queremos t_1, t_2 tq $\mathbb{P}(t_1 \leq T(x) \leq t_2) = 1 - \alpha = 0,95$ $(\alpha = 0,05)$

$$\Rightarrow \mathbb{P}(T(x) \leq t_2) - \mathbb{P}(T(X) \leq t_1) = 1 - \alpha$$

- Si usamos $\left.\begin{array}{l} t_2 = +\infty \\ t_1 = F_T^{-1}(\alpha) \end{array}\right\} \Rightarrow$ Nos da la igualdad deseada

Con esos valores
- $\mathbb{P}(T(x) \leq t_2) = \mathbb{P}(T(x) > 0) = 1$
- $\mathbb{P}(T(x) \leq t_1) = F_T(t_1) = F_T(F_T^{-1}(\alpha)) = \alpha$

- $t_1 \leq T(x) \leq t_2 \iff \dfrac{2\sum X_i}{t_2} \leq \theta \leq \dfrac{2\sum X_i}{t_1} \iff 0 < \theta \leq \dfrac{2\sum^n X_i}{F_T^{-1}(\alpha)}$ $\boxed{\text{R}}$

c) $\bar{X} = \frac{1}{n}\sum X_i \iff \sum X_i = n\bar{X}$

- Usando esto:

$$0 < \theta \leq \frac{2\cdot 7\cdot 4,77}{F_T^{-1}(0,05)}$$