Aux 12

P1 $\quad Y_i = \beta X_i + \varepsilon_i \; ; \; \varepsilon_i$ centrados, $Var(\varepsilon_i) = \sigma^2 \;\; \forall i$

$Cov(\varepsilon_i, \varepsilon_j) = 0 \quad \forall i \neq j$

a)

- $\tilde{\beta} = \frac{1}{n} \sum Y_i / X_i$

$$\Rightarrow \mathbb{E}(\tilde{\beta}) = \mathbb{E}\left(\frac{1}{n} \sum Y_i / X_i\right) = \frac{1}{n} \sum \frac{\mathbb{E}(Y_i)}{X_i} \overset{*}{=} \frac{1}{n} \sum \frac{\beta \cancel{X_i}}{\cancel{X_i}} = \beta \;//$$

$\tilde{\beta}$ insesgado

$$* \; \mathbb{E}(Y_i) = \mathbb{E}(\beta X_i + \varepsilon_i) = \mathbb{E}_{\beta}(\beta X_i) + \mathbb{E}(\varepsilon_i) = \beta X_i + 0 = \beta X_i$$

$$\bullet \ \operatorname{Var}(\tilde{\beta}) = \operatorname{Var}\left(\frac{1}{n}\sum \frac{y_i}{x_i}\right) = \frac{1}{n^2}\operatorname{Var}\left(\sum \frac{y_i}{x_i}\right)$$

$$= \frac{1}{n^2}\left[\sum_i \operatorname{Var}\left(\frac{y_i}{x_i}\right) + \sum_{i \neq j} \operatorname{Cov}\left(\frac{y_i}{x_i}, \frac{y_j}{x_j}\right)\right]$$

$$= \frac{1}{n^2}\left[\sum_i \frac{1}{x_i^2}\operatorname{Var}(y_i) + \sum_{i \neq j} \frac{1}{x_i}\cdot\frac{1}{x_j}\operatorname{Cov}(y_i, y_j)\right]$$

$$\overset{*}{=} \frac{1}{n^2}\left[\sigma^2 \sum_i 1/x_i^2 + 0\right] = \sigma^2/n^2 \sum_i 1/x_i^2 \quad //$$

$$* \ \operatorname{Var}(y_i) = \operatorname{Var}(\beta x_i + \varepsilon_i) = \operatorname{Var}(\varepsilon_i) = \sigma^2$$

$$* \ \operatorname{Cov}(y_i, y_j) = \operatorname{Cov}(\beta x_i + \varepsilon_i, \beta x_j + \varepsilon_j) = \operatorname{Cov}(\varepsilon_i, \varepsilon_j) = \sigma^2 \mathbb{1}_{\{i=j\}} = \delta_{ij}$$

b) $\hat{\beta} = (X^T X)^{-1} X^T Y$ cuando el modelo es $Y = X\beta + \varepsilon$

↳ se traduce en:

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{pmatrix}$$

$$\hat{\beta} = \left(\sum x_i^2\right)^{-1} \sum x_i y_i = \frac{\sum x_i y_i}{\sum x_i^2} \,//$$

- $$\mathbb{E}(\hat{\beta}) = \frac{1}{\sum x_i^2} \cdot \sum x_i \mathbb{E}(y_i) = \frac{\beta \cancel{\sum x_i^2}}{\cancel{\sum x_i^2}} = \beta \,//$$

$\hat{\beta}$ es insesgado

- $\mathrm{Var}(\hat{\beta}) = \sigma^2 (X^T X)^{-1} = \sigma^2 / \sum x_i^2$ //

c) Los compararemos según su varianza:

$\mathrm{Var}(\tilde{\beta}) = \sigma^2/n^2 \sum_{i=1}^{n} 1/x_i^2$ vs $\mathrm{Var}(\hat{\beta}) = \dfrac{\sigma^2}{\sum_{i=1}^{n} x_i^2}$

$\hookrightarrow$ Teo. GM: $\hat{\beta}$ es el mejor estimador (en el sentido de la varianza) entre los estimadores lineales e insesgados

- $\hat{\theta}$ es lineal si se escribe $\hat{\theta} = AY$, con A conveniente

- Si def $A = \frac{1}{n}\begin{pmatrix} 1/x_1 \\ \vdots \\ 1/x_n \end{pmatrix} \Rightarrow \tilde{\beta} = A^T y$

$\therefore$ $\tilde{\beta}$ es lineal, y de (c) sabemos que es insesgado

$\overset{TGM}{\Longrightarrow}$ $\hat{\beta}$ es mejor que $\tilde{\beta}$ en el sentido de la varianza $\blacksquare$

P2) ¿Cuál es el modelo lineal?

a) $Y_i = \vartheta_i + \varepsilon_i$, y queremos estimar $\vartheta = (\vartheta_1, \vartheta_2, \vartheta_3)$

$\Rightarrow$ En este caso $\beta_i = \vartheta_i$

$X_i = 1$

$\hookrightarrow$ El modelo matricial se $Y = X\beta + \varepsilon$

- Queremos buscar el EMC por def.

$$\min_{\vartheta} \sum_{i=1}^{3} (Y_i - \vartheta_i)^2 \iff \min_{\vartheta'} \overbrace{\sum_{i=1}^{2} (Y_i - \vartheta_i)^2 + (Y_3 - \pi + \vartheta_1 + \vartheta_2)^2}^{L(\vartheta)}$$

sa $\sum \vartheta_i = \pi$ $\qquad * \vartheta' = (\vartheta_1, \vartheta_2)$

- Como las funcs. son convexas c/r a ϑ_1, ϑ_2, ¡pido $\partial/\partial\theta_i = 0$!

- $$\frac{\partial L(\theta)}{\partial \vartheta_1} = -2(y_1 - \theta_1) + 2(y_3 - \pi + \vartheta_1 + \vartheta_2) \overset{!}{=} 0$$

$$\iff \quad 2\theta_1 + \vartheta_2 = \pi + y_1 - y_3 \quad (\text{ec } 1)$$

- $$\frac{\partial L(\theta)}{\partial \theta_2} = 0 \iff 2\theta_2 + \theta_1 = \pi + y_2 - y_3 \quad (\text{ec } 2)$$

Juntando ambos $\Rightarrow$

$$\hat{\theta}_1 = y_1 - (\bar{y} - \pi/3)$$

$$\hat{\theta}_2 = y_2 - (\bar{y} - \pi/3)$$

$$\hat{\theta}_3 = \pi - \hat{\theta}_1 - \hat{\theta}_2 = y_3 - (\bar{y} - \pi/3) \quad \blacksquare$$

b) Nos piden plantear el modelo paramétrico

$\Theta = (0, \pi)^3 \longrightarrow \vartheta = (\vartheta_1, \vartheta_2, \vartheta_3)$

$\mathcal{Y} = (0, \pi)^3$ o $\mathbb{R}^3$ (ambas tienen sentido)

- Podemos notar $Y_i \sim N(\vartheta_i, \sigma^2)$

$\mathcal{P} = \left\{ \prod_{i=1}^{3} f_i(y_i) \;\middle|\; f_i(y_i) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(y_i - \vartheta_i)^2} \right\}$

- Las hipótesis

$H_0: \vartheta_1 = \vartheta_2 = \vartheta_3 = \pi/3$ vs $H_1: \exists \vartheta_i$ tq $\vartheta_i \neq \pi/3$

Esto define

$\left. \begin{array}{l} \Theta_0 = \{\pi/3\}^3 \\ \Theta_1 = \Theta \setminus \Theta_0 \end{array} \right.$

c) Queremos encontrar $R_{TRV} = \{ y \mid \hat{\lambda}(y) \geq k_\alpha \}$

$$\hat{\lambda}(y) = \frac{\sup_{\theta \in \Theta} L(y,\theta)}{\sup_{\theta \in \Theta_0} L(y,\theta)} \overset{*}{=}$$

$$* \quad L(y,\theta) = \prod_{i=1}^{3} f_i(y_i) = \frac{1}{(2\pi)^{3/2}\sigma^3} e^{\frac{1}{2\sigma^2}\sum_{i=1}^{3}(y_i-\theta_i)^2}$$

$$L_7 \; \Theta_0 \quad \sup_{\theta \in \Theta_0} L(y,\theta) = L\left(y, \{\pi/3\}^3\right) = \frac{1}{(2\pi)^{3/2}\sigma^3} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{3}(y_i-\pi/3)}$$

$$\hookrightarrow \sup_{\theta \in \Theta} L(y,\theta) = \frac{1}{(2\pi)^{3/2}\sigma^3} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{3}(y_i - y_i + (\bar{y} - \pi/3))^2}$$

$$= \frac{1}{(2\pi)^{3/2}\sigma^3} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{3}(\bar{y} - \pi/3)^2}$$

$$\therefore \hat{\lambda}(y) = \frac{e^{-\frac{1}{2\sigma^2}\sum(\bar{y}-\pi/3)^2}}{e^{-\frac{1}{2\sigma^2}\sum(y_i-\pi/3)^2}} \overset{!}{\geq} K_\alpha$$

$$\Leftrightarrow \sum\left[(\bar{y}-\pi/3)^2 - (y_i - \pi/3)^2\right] \leq K_\alpha'$$

$$\Longleftrightarrow \quad \sum \left[\bar{y}^2 - 2\frac{\pi}{3}\bar{y} + \cancel{\left(\frac{\pi}{3}\right)^2} - y_i^2 + 2\frac{\pi}{3} y_i - \cancel{\left(\frac{\pi}{3}\right)^2} \right] \le K_\alpha'$$

$$\Longleftrightarrow \quad \sum \left[\bar{y}^2 - y_i^2 \right] - \cancel{2\pi\bar{y}} + \cancel{2\pi\bar{y}} \le K_\alpha'$$

$$\Longleftrightarrow \quad \sum \left(y_i^2 - \bar{y}^2 \right) \ge K_\alpha''$$