Aux 13

Identificamos componentes:

P1

$$\beta = \begin{pmatrix} \mu \\ \tau_1 \\ \vdots \\ \tau_r \end{pmatrix} \qquad Y = \begin{pmatrix} Y_{11} \\ Y_{12} \\ \vdots \\ Y_{1n} \\ Y_{21} \\ \vdots \\ Y_{rn} \end{pmatrix}$$

$\leftarrow$ Fila n

* Modelo

$$\boxed{Y = X\beta + \varepsilon}$$

donde

$$Y_{ij} = \mu + \tau_i + \varepsilon_{ij}$$

$$X = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{array}{l} \\ \\ \\ \leftarrow \text{fila } n \\ \\ \\ \\ \end{array}$$

• ¿Cuál es la dim(X)?

$X \in \mathbb{R}^{rn \times (r+1)}$

$X \in \mathbb{R}^{n \times m}$, $m < n$

$rg(X) = p < m \leftarrow$ Necesitan $m-p$ restricc.

* Por construcción, la matriz X tiene r bloques de n filas repetidas

$\Rightarrow$ Hay r filas que son li

$\Rightarrow rg(X) = r < r+1 \quad \therefore$ No es de rango completo

b) ∘ Queremos encontrar $\hat{\beta}_{MC}$ sujeto a $\sum_{i=1}^{r} \tau_i = 0$

↳ Planteamos MC:

$$\min \sum_{i,j} (y_{ij} - \mu - \tau_i)^2 =: f(\beta)$$

$$\text{sa} \quad \sum_i \tau_i = 0$$

$$* \sum_{i,j} (y_{ij} - (\mu + \tau_i))^2 = \sum_{i,j} y_{ij}^2 - 2 \sum_{i,j} y_{ij} (\mu + \tau_i) + \sum_{i,j} (\mu + \tau_i)^2$$

$$= \sum_{ij} y_{ij}^2 - 2\mu \sum_{ij} y_{ij} - 2 \sum_i \left(\tau_i \sum_j y_{ij}\right) + rn\mu^2 + 2n\mu \sum_i \tau_i$$

$$+ n \sum_i \tau_i^2$$

$$\therefore\ f(\beta) = \sum_{ij} y_{ij}^2 - 2\mu \sum_{ij} y_{ij} - 2\sum_i \left(\tau_i \sum_j y_{ij}\right) + rn\mu^2 + \cancel{2n\mu \sum_i \tau_i}^{\,0} + n\sum_i \tau_i^2$$

↳ Uso la restricción

$$\Rightarrow \boxed{f(\beta) = \sum_{i,j} y_{ij}^2 - 2\mu \sum_{i,j} y_{ij} - 2\sum_i \left(\tau_i \sum_j y_{ij}\right) + rn\mu^2 + n\sum_i \tau_i^2}$$

Plantear el KKT:

$$L(\beta) = f(\beta) + \lambda h(\beta) \quad , \quad \text{con } \boxed{h(\beta) = \sum_i \tau_i}$$

$$\Rightarrow \nabla L(\beta) = \underline{\nabla f(\beta)} + \lambda \underline{\nabla h(\beta)} = 0$$

$$\lambda \geq 0$$

$\frac{\partial h}{\partial \mu} = 0$ $\quad$ $\frac{\partial h}{\partial \tau_i} = 1 \quad \forall i \in \{1 \dots r\}$

$\frac{\partial f}{\partial \mu} = -2 \sum_{i,j} y_{ij} + 2rn\mu$

$\Rightarrow \frac{\partial L}{\partial \mu} = 0 \iff \frac{\partial f}{\partial \mu} = 0$

$\iff \left[\hat{\mu} = \frac{\sum_{i,j} y_{ij}}{rn} = \overline{y} \right]$

$$\frac{\partial f}{\partial \tau_i} = -2 \sum_j y_{ij} + 2n\tau_i$$

$$\Rightarrow \frac{\partial L}{\partial \tau_i} = \frac{\partial f}{\partial \tau_i} + \lambda \frac{\partial h}{\partial \tau_i} \overset{!}{=} 0$$

$$\Leftrightarrow -2\sum_j y_{ij} + 2n\tau_i + \lambda = 0$$

$$\Leftrightarrow \tau_i = \frac{\sum_j y_{ij}}{n} - \frac{\lambda}{2n} = \overline{y_i} - \frac{\lambda}{2n}$$

¿Cuál es el valor de λ?

Recordamos $\sum \tau_i = 0$

$$\Rightarrow \sum_i \tau_i = \sum_i \left(\frac{1}{n} \sum_j y_{ij} - \frac{\lambda}{2n} \right) = \frac{1}{n} \sum_{i,j} y_{ij} - \frac{\lambda r}{2n} \overset{!!}{=} 0$$

$$\Rightarrow \frac{\lambda}{2n} = \frac{1}{nr} \sum_{i,j} y_{ij} = \bar{y}$$

$$\therefore \left[\hat{\tau}_i = \bar{y}_i - \lambda/2n = \bar{y}_i - \bar{y} \right]$$

P2) $Y = X\beta + \varepsilon$

Extendido $Y = X_{\cdot,1}\beta_1 + X_{\cdot,2}\beta_2 + \dots + X_{\cdot,p}\beta_3$

↳ Suponga que olvidamos algunos $X_{\cdot,k}$; con $k \in \{1 \dots p\}$

⇒ Nuevo modelo $Y = X_{\cdot,1}\beta_1 + \dots + X_{\cdot,k-1}\beta_{k-1} + X_{\cdot,k+1}\beta_{k+1} + \dots + X_{\cdot,p}\beta_p$

⇒ En definitiva, lo que pasa es que borremos algunas columnas de X

- ¿Cómo se vería el estimador antes de olvidar variables?

$$\hat{\beta}_{MC} = (X^T X)^{-1} X^T y \implies (X^T X)_{jk} = \left(\sum_{i=1}^{n} X_{ij} X_{ik} \right)_{jk}$$

- $$X = \begin{bmatrix} X_{11} & X_{12} & \dots & X_{1p} \\ X_{21} & X_{22} & \dots & X_{2p} \\ & & \ddots & \\ X_{n1} & & \dots & X_{np} \end{bmatrix}$$

↳ Es una matriz diagonal

$$X^T X = \begin{pmatrix} \sum_{i=1}^{n} X_{i1}^2 & & & 0 \\ & \sum_{i=1}^{n} X_{i2}^2 & & \\ & & \ddots & \\ 0 & & & \sum_{i=1}^{n} X_{ip}^2 \end{pmatrix}$$

$$\Rightarrow \quad (X^TX)^{-1} = \begin{pmatrix} 1/\sum_{i=1}^{n} x_{i1}^2 & & 0 \\ & \ddots & \\ 0 & & 1/\sum_{i=1}^{n} x_{ip}^2 \end{pmatrix}$$

$$* \quad X^Ty = \begin{pmatrix} \sum_{i=1}^{n} x_{i1} y_i \\ \vdots \\ \sum_{i=1}^{n} x_{ip} y_i \end{pmatrix}$$

$$\hat{\beta}_{MC} = \left(\frac{\sum_{i=1}^{n} X_{ij}\, y_i}{\sum_{i=1}^{n} X_{ij}^2} \right)_{j=1}^{p}$$

$$* \; y_i = X_{i,\cdot}\,\beta + \varepsilon_i$$

↳ Veamos que $\hat{\beta}_{MC}$ es insesgado:

$$\mathbb{E}(\hat{\beta}_{MC,j}) = \frac{1}{\sum_{i=1}^{n} X_{ij}^2} \cdot \sum_{i=1}^{n} X_{ij}\, \mathbb{E}(y_i) = \frac{\sum_{i=1}^{n} X_{ij} \left(\sum_{k=1}^{p} X_{ik}\beta_k \right)}{\sum_{i=1}^{n} X_{ij}^2}$$

$$\therefore \mathbb{E}(\hat{\beta}_{MC_j}) = \frac{\sum_{i=1}^{n} X_{ij}\left(\sum_{k=1}^{\cancel{p}\,l} X_{ik}\beta_k\right)}{\sum_{i=1}^{n} X_{ij}^2} = \frac{\sum_{k=1}^{\cancel{p}\,l} \beta_k \sum_{i=1}^{n} X_{ij} X_{ik}}{\sum_{i=1}^{n} X_{ij}^2}$$

$l < p$

$j = 1 \dots p$

$\hookrightarrow j \in \{1 \dots l\}$

Hip

$$\hookrightarrow = \frac{\beta_j \cancel{\sum_{i=1}^{n} X_{ij}^2}}{\cancel{\sum_{i=1}^{n} X_{ij}^2}} = \beta_j \qquad \therefore \hat{\beta}_{MC} \text{ es insesgado}$$

$\therefore$ Por la explicación conversada, no es relevante eliminar columnas de X

B3) Como olvide las variables X_2, mi "nuevo modelo" es:

$$Y = X_1\beta_1 + \varepsilon' \quad ; \quad \varepsilon' \text{ error usual}$$

$$\Rightarrow \hat{\beta}_1 = (X_1^T X_1)^{-1} X_1^T Y$$

$$\hookrightarrow \mathbb{E}(\hat{\beta}_1) = (X_1^T X_1)^{-1} X_1^T \mathbb{E}(Y)$$

$$= (X_1^T X_1)^{-1} X_1^T (X_1\beta_1 + X_2\beta_2)$$

$$= \beta_1 + (X_1^T X_1)^{-1} X_1^T X_2 \beta_2 \quad \therefore \hat{\beta}_1 \text{ es sesgado} \quad //$$

- Planteamos modelo $Z = X_{1(\cdot,j)} = X_2\beta_2 + \varphi_j$

$\Rightarrow$ Sabemos $\hat{\beta}_2 = (X_2^T X_2)^{-1} X_2^T X_{1(\cdot,j)}$

$\ast\ \hat{Z} = X_2\hat{\beta}_2$

$$\Rightarrow \hat{\varphi}_j = Z - \hat{Z} = Z - X_2\hat{\beta}_2$$

$$= Z - X_2(X_2^T X_2)^{-1} X_2^T Z$$

$$= \left(I - X_2(X_2^T X_2)^{-1} X_2^T\right) Z$$

$$= \left(I - X_2(X_2^T X_2)^{-1} X_2^T\right) X_{1(\cdot,j)}$$

$$\Rightarrow \boxed{X_j^* = \left(I - X_2(X_2^T X_2)^{-1} X_2^T\right) X_1}$$

- Planteo modelo final $y = X_j^* \beta_1 + \delta$

$$\therefore \tilde{\beta}_1 = (X_j^{*T} X_j^*)^{-1} X_j^{*T} y$$

$$\hookrightarrow \mathbb{E}(\tilde{\beta}_1) = (X_j^{*T} X_j^*)^{-1} X_j^{*T} \mathbb{E}(y)$$

$$= (X_j^{*T} X_j^*)^{-1} X_j^{*T} (X_1 \beta_1 + X_2 \beta_2)$$

$$= \underbrace{(X_j^{*T} X_j^*)^{-1} X_j^{*T} X_1 \beta_1}_{(1)} + \underbrace{(X_j^{*T} X_j^*)^{-1} X_j^{*T} X_2 \beta_2}_{(2)}$$

$\boxed{(2)}$ En (2) tenemos:

$$X_1^{*T} X_2 = X_1^T \left(I - X_2 (X_2^T X_2)^{-1} X_2^T \right)^T X_2$$

$$= X_1^T \left(I - X_2 (X_2^T X_2)^{-1} X_2^T \right) X_2$$

$$= X_1^T (X_2 - X_2) = 0$$

$\therefore \boxed{(2) = 0}$

(1) Calculemos (1):

$\circ \ (X_1^{*T} X_1^*)^{-1} X_1^{*T} X_1 \beta_1$

(3)

$$= \left[X_1^T \left(I - X_2 (X_2^T X_2)^{-1} X_2^T \right) \circ \left(I - X_2 (X_2^T X_2)^{-1} X_2^T \right) X_1 \right]^{-1}$$

$$\circ \ X_1^T \left(I - X_2 (X_2^T X_2)^{-1} X_2^T \right) X_1 \beta_1$$

$$(3) = I - 2 X_2 (X_2^T X_2)^{-1} X_2^T + X_2 \cancel{(X_2^T X_2)^{-1} X_2^T X_2} (X_2^T X_2)^{-1} X_2^T$$

$$= I - X_2 (X_2^T X_2)^{-1} X_2^T \ //$$

$\therefore$ Volviendo a (1):

$$(1) = \left[\cancel{X_1^T\left(I - X_2(X_2^TX_2)^{-1}X_2^T\right)X_1}\right]^{-1}\left[\cancel{X_1^T\left(I - X_2(X_2^TX_2)^{-1}X_2^T\right)X_1}\,\beta_1\right]$$

$$= \beta_1$$

$$\therefore \; E(\tilde{\beta}_1) = (1) + (2) = \beta_1$$

$\therefore$ $\tilde{\beta}_1$ es insesgado $\blacksquare$