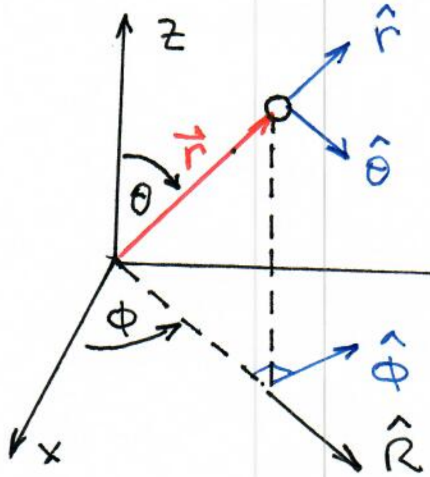
MECÁNICA OTO 2021 S4 P. ACEITUNO 25/3/21

COORDENADAS ESFÉRICAS



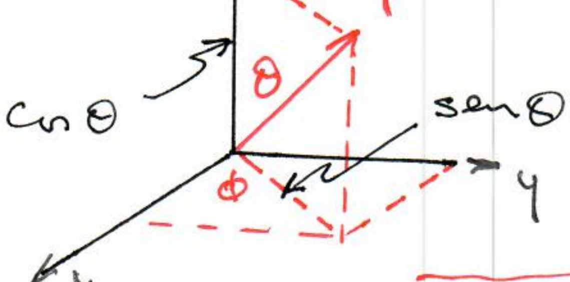
POSICIÓN

$$\vec r = r\,\hat r$$

PARA CADA VALOR DE r, ϕ, θ HAY SOLO UN PUNTO

$$r \in [0, \infty)$$
$$\theta \in [0, \pi]$$
$$\phi \in [0, 2\pi]$$

LA DIRECCION DE $\hat r$ SE PUEDE EXPLICITAR UTILIZANDO LA REPRESENTACIÓN CARTESIANA



SEGÚN $\hat\imath$) $\operatorname{sen}\theta\cos\phi$

SEGÚN $\hat\jmath$) $\operatorname{sen}\theta\operatorname{sen}\phi$

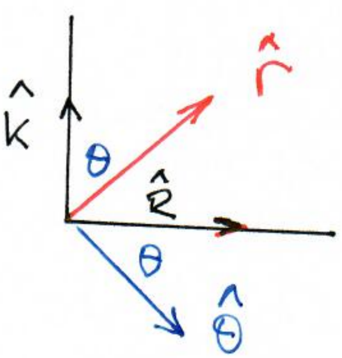
SEGÚN $\hat k$) $\cos\theta$

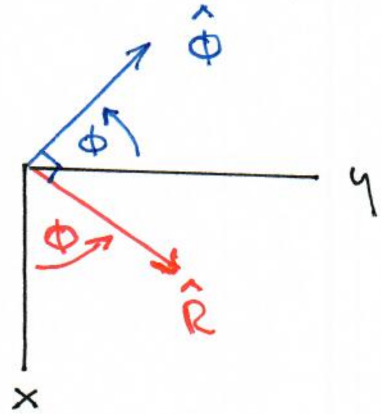
$$\vec r = r\operatorname{sen}\theta\cos\phi\,\hat\imath + r\operatorname{sen}\theta\operatorname{sen}\phi\,\hat\jmath + r\cos\theta\,\hat k$$

LAS DIRECCIONES PRINCIPALES DEL SIST. DE COORD. ESFÉRICAS ESTÁN DEFINIDAS POR LOS VECTORES UNITARIOS $\hat{r}, \hat{\theta}, \phi$

$\hat{\theta}$ CONTENIDA EN EL PLANO $(\hat{k}, \hat{r})$ PERPENDICULAR A $\hat{r}$ EN EL SENTIDO DE θ CRECIENTE

$\hat{\phi}$ CONTENIDA EN EL PLANO $(\hat{i}, \hat{j})$ PERPENDICULAR A $\hat{r}$ Y A $\hat{\theta}$ EN EL SENTIDO DE ϕ CRECIENTE





$$\hat{R} = \cos\phi\,\hat{i} + \operatorname{sen}\phi\,\hat{j}$$

$$\hat{r} = \cos\theta\,\hat{k} + \operatorname{sen}\theta\,\hat{R}$$

$$\hat{r} = \operatorname{sen}\theta\cos\phi\,\hat{i} + \operatorname{sen}\theta\operatorname{sen}\phi\,\hat{j} + \cos\overset{\theta}{\cancel{\phi}}\,\hat{k}$$

$$\hat{\theta} = -\operatorname{sen}\theta\,\hat{k} + \cos\theta\,\hat{R}$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{i} + \cos\theta\operatorname{sen}\phi\,\hat{j} - \operatorname{sen}\theta\,\hat{k}$$

$$\hat{\phi} = -\operatorname{sen}\phi\,\hat{i} + \cos\phi\,\hat{j}$$

DE LAS ECUACIONES ANTERIORES SE INFIERE QUE...

$$\begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} = \underbrace{\begin{bmatrix} \operatorname{sen}\theta\cos\phi & \operatorname{sen}\theta\operatorname{sen}\phi & \cos\overset{\theta}{\cancel{\phi}} \\ \cos\theta\cos\phi & \cos\theta\operatorname{sen}\phi & -\operatorname{sen}\theta \\ -\operatorname{sen}\phi & \cos\phi & 0 \end{bmatrix}}_{A\ :\ \text{MATRIZ DE TRANSFORMACIÓN}} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$

$$\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = A^{-1} \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} \qquad (*)$$

UTILIZANDO ESTA APROXIMACIÓN

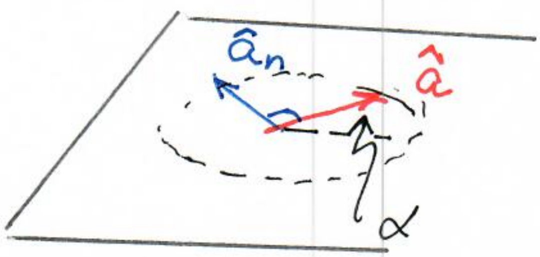
$$\vec{v} = \frac{d}{dt}\left[r\operatorname{sen}\theta\cos\phi\,\hat{i} + r\operatorname{sen}\theta\operatorname{sen}\phi\,\hat{j} + r\cos\theta\,\hat{k} \right]$$

$$\vec{v} = \begin{cases} \hat{i}\big)\ \dot{r}\operatorname{sen}\theta\cos\phi + r\dot{\theta}\cos\theta\cos\phi - r\dot{\phi}\operatorname{sen}\theta\operatorname{sen}\phi \\ \hat{j}\big)\ \dot{r}\operatorname{sen}\theta\operatorname{sen}\phi + r\dot{\theta}\cos\theta\operatorname{sen}\phi + r\operatorname{sen}\theta\,\dot{\phi}\cos\phi \\ \hat{k}\big)\ \dot{r}\cos\theta - r\dot{\theta}\operatorname{sen}\theta \end{cases}$$

DE LA ECUACIÓN (*) SE PUEDE ENCONTRAR $\vec{v} = f(r, \theta, \phi, \hat{r}, \hat{\theta}, \hat{\phi})$

UN PROCEDIMIENTO SIMILAR PERO AÚN MÁS ENGORROSO PERMITE ENCONTRAR UNA EXPRESIÓN PARA LA ACELERACIÓN

OTRO METODO MAS SIMPLE SOLO REQUIERE TENER EN CUENTA LO SIGUIENTE



$\hat{a}$ VECTOR UNITARIO QUE ROTA EN UN PLANO

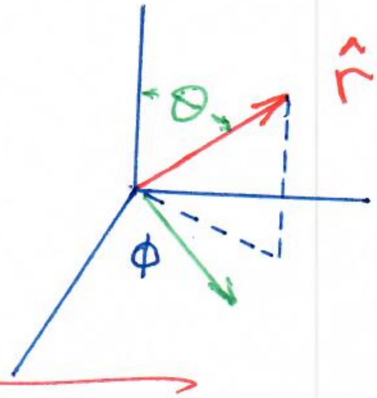
$$\frac{d\hat{a}}{dt} = \frac{d\hat{a}}{d\alpha}\frac{d\alpha}{dt} = \hat{a}_n\,\dot{\alpha}$$

$\hat{a}_n \perp \hat{a}$ EN EL SENTIDO DE α CRECIENTE

ESTO YA LO VIMOS EN COORDENADAS POLARES CUANDO SE CALCULA $\frac{d\hat{\rho}}{dt} = \dot{\theta}\hat{\theta}$

$$\star \quad \boxed{\frac{d\hat{a}}{d\alpha} = \hat{a}_n}$$

- Posición $\hat{r} = r\,\hat{r}$



LA DIRECCIÓN DE $\hat{r}$ DEPENDE DE θ y de ϕ, PERO NO DE r

$\boxed{\text{NOTA:}}$

SEA $z = f(x, y, w)$

y x, y, w DEPENDEN DE t

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial w}\frac{dw}{dt}$$

Ej

$$z = x y^2 w^3 \qquad \frac{\partial f}{\partial x} = y^2 w^3$$

$$x = at \qquad \frac{\partial f}{\partial y} = 2xyw^3$$

$$y = b t^{-1}$$

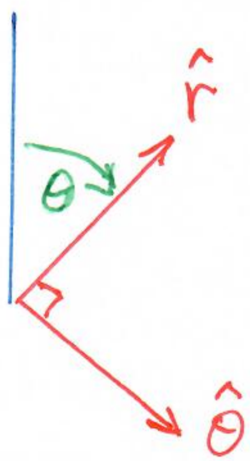
$$w = \operatorname{sen} t \qquad \frac{\partial f}{\partial w} = 3xy^2w^2$$

$$\frac{dz}{dt} = y^2 w^3 \cdot a + 2xyw^3\left(-\frac{b}{t^2}\right) + 3xy^2w^2 \cos t$$

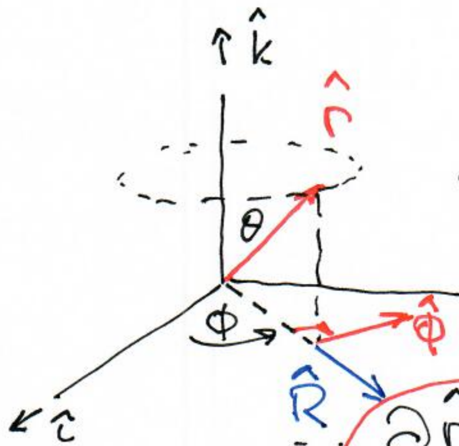
REEMPLAZANDO x, y, w POR SUS EXPRESIONES EN FUNCIÓN DEL TIEMPO SE ENCUENTRA dz/dt

$$\vec{v} = \frac{d}{dt}(r\hat{r}) = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt}$$

$$\frac{d\hat{r}}{dt} = \frac{\partial \hat{r}}{\partial \theta}\dot{\theta} + \frac{\partial \hat{r}}{\partial \phi}\dot{\phi}$$



$$\frac{\partial \hat{r}}{\partial \theta} = \hat{\theta}$$



AL CAMBIAR ϕ $\hat{r}$ SE MUEVE SOBRE UN <u>CONO</u>

$$\frac{\partial \hat{r}}{\partial \phi} \neq \hat{\phi} \quad \star$$

PERO $\hat{r} = \cos\theta\,\hat{k} + \operatorname{sen}\theta\,\hat{R}$

$$\frac{\partial \hat{r}}{\partial \phi} = \operatorname{sen}\theta\,\frac{\partial \hat{R}}{\partial \phi} = \operatorname{sen}\theta\,\hat{\phi}$$

$$\frac{\partial \hat{r}}{\partial \phi} = \operatorname{sen}\theta\,\hat{\phi}$$

JUNTANDO LOS TÉRMINOS

$$\vec{v} = \dot{r}\hat{r} + r\left[\dot{\theta}\hat{\theta} + \operatorname{sen}\theta\,\dot{\phi}\,\hat{\phi}\right]$$

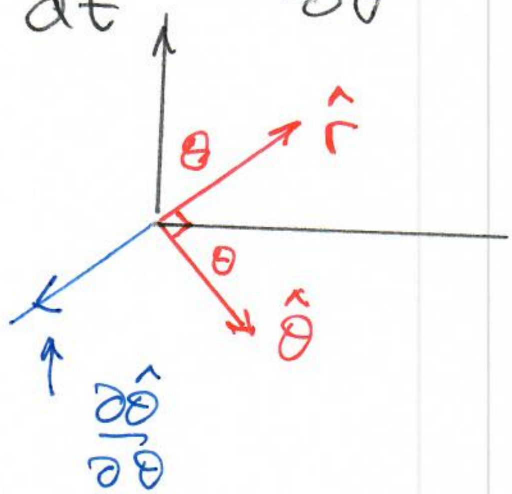
$$\vec{v} = \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta} + r\dot{\phi}\,\mathrm{sen}\,\theta\,\hat{\phi}$$

ACELERACIÓN

SE REQUIERE ENCONTRAR EXPRESIONES PARA $\dfrac{d\hat{\theta}}{dt}$ y $\dfrac{d\hat{\phi}}{dt}$

AL IGUAL QUE $\hat{r}$, $\hat{\theta}$ DEPENDE DE θ y ϕ

$$\frac{d\hat{\theta}}{dt} = \frac{\partial\hat{\theta}}{\partial\theta}\dot{\theta} + \frac{\partial\hat{\theta}}{\partial\phi}\dot{\phi}$$



$$\frac{\partial\hat{r}}{\partial\theta} = \hat{\theta} \longrightarrow$$

$$\longrightarrow \frac{\partial\hat{\theta}}{\partial\theta} = -\hat{r}$$

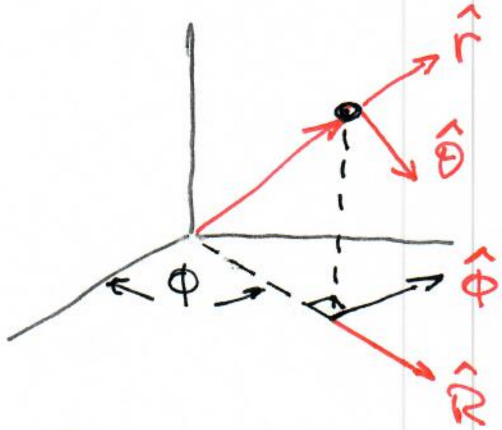
$$\hat{\theta} = \cos\theta\,\hat{R} - \mathrm{sen}\,\theta\,\hat{k}$$

$$\frac{\partial\hat{\theta}}{\partial\phi} = \cos\theta\,\underbrace{\frac{\partial\hat{R}}{\partial\phi}}_{\hat{\phi}} = \cos\theta\,\hat{\phi}$$

$$\boxed{\frac{d\hat{\theta}}{dt} = -\dot{\theta}\,\hat{r} + \dot{\phi}\cos\theta\,\hat{\phi}}$$

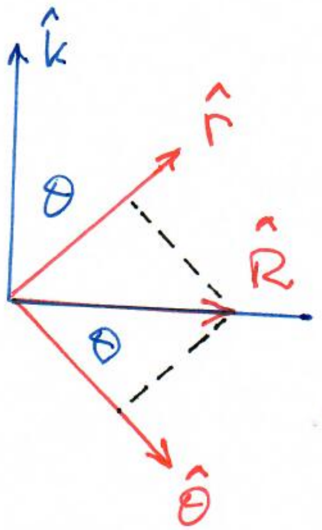
EL VECTOR $\hat{\phi}$ ESTÁ EN EL PLANO (X-Y)

NO DEPENDE DE r NI DE θ



$$\frac{d\hat{\phi}}{dt} = \frac{d\hat{\phi}}{d\phi}\,\dot{\phi}$$

$$\frac{d\hat{R}}{d\phi} = \hat{\phi} \longrightarrow \frac{d\hat{\phi}}{d\phi} = -\hat{R}$$



$$\hat{R} = \operatorname{sen}\theta\,\hat{r} + \cos\theta\,\hat{\theta}$$

$$\boxed{\frac{d\hat{\phi}}{dt} = -\dot{\phi}\operatorname{sen}\theta\,\hat{r} - \dot{\phi}\cos\theta\,\hat{\theta}}$$

$$\vec{v} = \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta} + r\dot{\phi}\operatorname{sen}\theta\,\hat{\phi}$$

$$\vec{a} = \ddot{r}\,\hat{r} + \dot{r}\,\underline{\frac{d\hat{r}}{dt}} + \dot{r}\dot{\theta}\,\hat{\theta} + r\ddot{\theta}\,\hat{\theta} + r\dot{\theta}\,\underline{\frac{d\hat{\theta}}{dt}}$$

$$+ \dot{r}\dot{\phi}\operatorname{sen}\theta\,\hat{\phi} + r\ddot{\phi}\operatorname{sen}\theta\,\hat{\phi} + r\dot{\phi}\dot{\theta}\cos\theta\,\hat{\phi}$$

$$+ r\dot{\phi}\operatorname{sen}\theta\,\frac{d\hat{\phi}}{dt}$$

$$\vec{a} = a_r \hat{r} + a_\theta \hat{\theta} + a_\phi \hat{\phi}$$

$$a_r = \ddot{r} - r\dot{\phi}^2 \operatorname{sen}^2\theta - r\dot{\theta}^2$$

$$a_\theta = r\ddot{\theta} - r\dot{\phi}^2 \operatorname{sen}\theta \cos\theta + 2\dot{r}\dot{\theta}$$

$$a_\phi = 2\dot{r}\dot{\phi} \operatorname{sen}\theta + r\ddot{\phi} \operatorname{sen}\theta + 2r\dot{\theta}\dot{\phi}\cos\theta$$