OSCILACIONES

- movimiento periódico en torno a algún equilibrio debido a que, al salir del equilibrio, hay fuerzas que devuelven a este estado
- puede haber pérdida de energía (amortiguamiento)
- " " inyección " (forzamiento)

- libres, amortiguados o forzados

- partícula m sujeta a fuerza $\vec{F}(\vec{r},\vec{v},t)$
- equilibrio en posición $\vec{r}_{eq} \Rightarrow \vec{F}(\vec{r}_{eq})=\vec{0}$
- aprox. lineal $\vec{F}=-k(\vec{r}-\vec{r}_{eq})$

  $\hookrightarrow \vec{r}_{eq}=\vec{0} \Rightarrow \vec{F}=-k\vec{r}$

Oscilación libre

$$m\vec{a}=-k\vec{r}$$

1D

$$m\ddot{x}=-kx$$

$\omega^2 \equiv k/m$

$$\ddot{x}+\omega^2 x=0$$

edo lineal $\longrightarrow x=Ce^{\lambda t} \longrightarrow \ddot{x}=\lambda^2 Ce^{\lambda t}$

$C=|C|e^{i\delta}$

$$\lambda^2+\omega^2=0$$

$$\lambda=\pm i\omega$$

$$x=C_1e^{i\omega t}+C_2e^{-i\omega t}$$

$\longrightarrow C_1,C_2 \in \mathbb{C} \longrightarrow C_1=Ae^{i\delta} \quad C_2=Be^{i\delta}$

$$
\begin{aligned}
x&=e^{i\delta}(Ae^{i\omega t}+Be^{-i\omega t})\\
&=e^{i\delta}(A\cos\omega t+iA\sin\omega t+B\cos\omega t-iB\sin\omega t)\\
&=e^{i\delta}[(A+B)\cos\omega t+i(A-B)\sin\omega t)\\
&=e^{i\delta}(\bar{A}\cos\omega t+i\bar{B}\sin\omega t)\\
&=(\cos\delta+i\sin\delta)(\bar{A}\cos\omega t+i\bar{B}\sin\omega t)\\
&=\bar{A}\cos\delta\cos\omega t+i\bar{B}\cos\delta\sin\omega t+i\bar{A}\sin\delta\cos\omega t-\bar{B}\sin\delta\sin\omega t
\end{aligned}
$$

Re

$$x=\bar{A}\cos\delta\cos\omega t-\bar{B}\sin\delta\sin\omega t$$

$$\dot{x}=-\omega\bar{A}\cos\delta\sin\omega t-\omega\bar{B}\sin\delta\cos\omega t$$

$$\begin{cases} x_0=x(0)=\bar{A}\cos\delta \\ \dot{x}_0=\dot{x}(0)=-\omega\bar{B}\sin\delta \end{cases}$$

$$\boxed{x=x_0\cos\omega t+\frac{\dot{x}_0}{\omega}\sin\omega t}$$

Oscilación amortiguada $\quad \vec{F} \propto \vec{v}$

$$m\vec{a} = -c\vec{v} - k(\vec{r} - \vec{r}_{eq}) \qquad \Big/ \quad \vec{r} - \vec{r}_{eq} = \vec{x}$$

$$m\vec{a} = -c\vec{v} - k\vec{x}$$

$$\ddot{x} = -\frac{c}{m}\dot{x} - \frac{k}{m}x \qquad b \equiv \frac{c}{m}$$

$$\ddot{x} + b\dot{x} + \omega^2 x = 0$$

edo lineal $x = Ae^{\lambda t}$

$$\lambda^2 + b\lambda + \omega^2 = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4\omega^2}}{2} \qquad b = 2\gamma$$

$$\lambda = -\gamma \pm \sqrt{\gamma^2 - \omega^2}$$

$\omega = \gamma$: amortiguamiento crítico

$$x = Ae^{-\gamma t}$$

$\omega > \gamma$: amortiguamiento subcrítico

$$\gamma^2 - \omega^2 < 0 \longrightarrow \lambda = \gamma \pm \omega\sqrt{(\gamma/\omega)^2 - 1} = \gamma \pm i\underbrace{\omega\sqrt{1 - (\gamma/\omega)^2}}_{\Omega < \omega}$$

$$x = e^{-\gamma t}(Ae^{i\Omega t} + Be^{-i\Omega t})$$

$\omega < \gamma$: amortiguamiento supercrítico

$$\gamma^2 - \omega^2 > 0 \longrightarrow \lambda = \gamma \pm \underbrace{\gamma\sqrt{1 - (\omega/\gamma)^2}}_{\Gamma < \gamma}$$

$$x = e^{-\gamma t}(Ae^{\Gamma t} + Be^{-\Gamma t})$$

Oscilación forzada

- ej: forzamiento periódico (sin amortig.)

$$\ddot{x} + \omega_0^2 x = f(t) = f_0 \sin \omega t$$

$$\hookrightarrow x = x_h + x_p$$

sol. homo. $\rightarrow$ $x_h = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}$

sol. part. $\rightarrow$ $x_p = C\sin\omega t$ $\rightarrow$ $\ddot{x}_p = -\omega^2 C\sin\omega t$

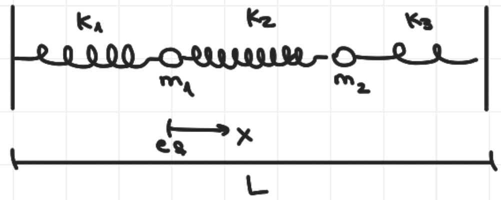
$$-\omega^2 C + \omega_0^2 C = f_0$$

$$C(\omega_0^2 - \omega^2) = f_0$$

$$x(t) = Ae^{i\omega t} + Be^{-i\omega t} + \frac{f_0}{\omega_0^2 - \omega^2}\sin\omega t$$

resonancia

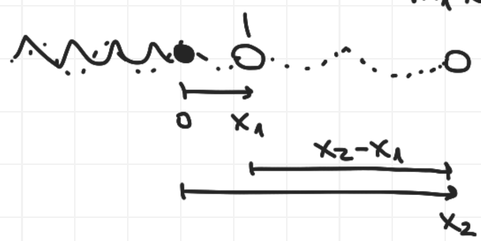
P2



a)

mov. 1: $m_1\ddot{x}_1 = -k_1x_1 + k_2(x_2-x_1)$

$$\longrightarrow \quad m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = 0 \qquad (1)$$



mov. 2: $m_2\ddot{x}_2 = -k_3x_2 + k_2(x_2-x_1)$

$$\longrightarrow \quad m_2\ddot{x}_2 + (k_2+k_3)x_2 - k_2x_1 = 0$$

sist. acoplado de 2×2:

$$m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = 0$$
$$m_2\ddot{x}_2 + (k_2+k_3)x_2 - k_2x_1 = 0$$

$$\begin{pmatrix} m_1\ddot{x}_1 & + & (k_1+k_2)x_1 - k_2x_2 \\ m_2\ddot{x}_2 & + & (k_2+k_3)x_2 - k_2x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} m_1\ddot{x}_1 \\ m_2\ddot{x}_2 \end{pmatrix} + \begin{pmatrix} (k_1+k_2)x_1 & -k_2x_2 \\ -k_2x_1 & +(k_2+k_3)x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}}_{M}\underbrace{\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix}}_{\ddot{\vec{x}}} + \underbrace{\begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{pmatrix}}_{K}\underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\vec{x}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$M\ddot{\vec{x}} + K\vec{x} = \vec{0}$$ — · ec. de mov. vectorial (lineal)
· similitud con oscilación 1D

ansatz: $\vec{z} = \vec{a}e^{i\omega t} \rightarrow \ddot{\vec{z}} = -\omega^2\vec{a}\,e^{i\omega t}$

ω desconocido

$$M\cdot -\omega^2\vec{a}e^{i\omega t} + K\vec{a}e^{i\omega t} = 0$$

$$(K-\omega^2 M)\vec{a} = 0 \quad \rightarrow \text{ ec de valores propios.}$$

$$\det(K-\omega^2 M) = 0 \quad \text{entrega los valores propios } \omega$$

b) $k_1 = k_2 = k_3 = k$

$m_1 = m_2 = m$

$$K-\omega^2 M = \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix} - \omega^2 \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

$$= \begin{pmatrix} 2k-\omega^2 m & -k \\ -k & 2k-\omega^2 m \end{pmatrix}$$

$$|K-\omega^2 M| = (2k-\omega^2 m)^2 - k^2 \overset{!}{=} 0$$

$$2k-\omega^2 m = \pm k$$

$$\omega^2 m = 2k \pm k$$

$$\omega^2 = \frac{2k\pm k}{m}$$

$$\omega_1 = \sqrt{\frac{k}{m}} \quad \text{y} \quad \omega_2 = \sqrt{\frac{3k}{m}} \quad \rightarrow \text{ frecuencias normales de oscilación}$$

c) modos normales $\rightarrow$ vectores propios (normalizados a 1)

$$(K-\omega^2 M)\vec{a} = 0$$

$\omega_1^2 = k/m$

$$\begin{pmatrix} 2k-k & -k \\ -k & 2k-k \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

$$k\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \longrightarrow a_1 - a_2 = 0 \rightarrow a_1 = a_2$$

$$\vec{a}(\omega_1) = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_1 \end{pmatrix} = a_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} = A_1 \cdot \underbrace{\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}}_{\hat{a}_1}$$

$\omega_2^2 = 3k/m$

$$\begin{pmatrix} 2k-3k & -k \\ -k & 2k-3k \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

$$-k \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \longrightarrow a_1 + a_2 = 0 \rightarrow a_2 = -a_1$$

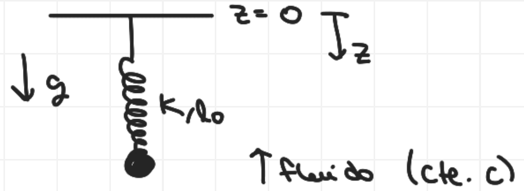
$$\vec{a}(\omega_2) = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ -a_1 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = A_2 \underbrace{\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}}_{\hat{a}_2}$$

$$\vec{z} = A_1 \hat{a}_1 e^{i\omega_1 t} + A_2 \hat{a}_2 e^{i\omega_2 t}$$

$$\vec{x} = \mathrm{Re}\,\vec{z} \qquad (A = Be^{i\phi})$$

$$\vec{x}(t) = B_1 \hat{a}_1 \cos(\omega_1 t + \phi_1) + B_2 \hat{a}_2 \cos(\omega_2 t + \phi_2)$$

P1



a) · ec. de mov. desde equilibrio. considerar sol. sobreamortiguada.

$$m\ddot{z} = -k(z-l_0) - c\dot{z} + mg$$

$$\ddot{z} = -\frac{k}{m}(z-l_0) - \frac{c}{m}\dot{z} + g \qquad b \equiv c/m \qquad \omega^2 \equiv k/m$$

$$\ddot{z} = -\omega^2(z-l_0) - b\dot{z} + g$$

$$\ddot{z} = -\omega^2\left(z - l_0 - \frac{g}{\omega^2}\right) - b\dot{z} \qquad \Big|\; y = z - \left(l_0 + \frac{g}{\omega^2}\right)$$

$$\hookrightarrow \dot{y} = \dot{z}$$

$$\ddot{y} = -\omega^2 y - b\dot{y}$$

$$\ddot{y} + b\dot{y} + \omega^2 y = 0$$

$$y(t) = e^{-\gamma t}\left(Ae^{\Gamma t} + Be^{-\Gamma t}\right) \qquad \text{con} \quad 2\gamma = b \quad \text{y} \quad \Gamma = \gamma\sqrt{1-\left(\frac{\omega}{\gamma}\right)^2}$$

b) $y(0) = H$, $\dot{y}(0) = v_0$

$$y(0) = A + B = H \longrightarrow B = H - A$$

$$\dot{y} = -\gamma e^{-\gamma t}\left(Ae^{\Gamma t} + Be^{-\Gamma t}\right) + e^{-\gamma t}\left(A\Gamma e^{\Gamma t} - B\Gamma e^{-\Gamma t}\right)$$

$$\dot{y}(0) = -\gamma(A+B) + \Gamma(A-B) = v_0$$

$$-A(\gamma-\Gamma) - B(\gamma+\Gamma) = v_0$$

$$-A(\gamma-\Gamma) - (H-A)(\gamma+\Gamma) = v_0$$

$$-A(\gamma-\Gamma) - H(\gamma+\Gamma) + A(\gamma+\Gamma) = v_0$$

$$A(\gamma+\Gamma-\gamma+\Gamma) = v_0 + H(\gamma+\Gamma)$$

$$\boxed{A = \frac{v_0 + H(\gamma+\Gamma)}{2\Gamma}}$$

$$B = H - \frac{V_0 + H(\gamma+\Gamma)}{2\Gamma}$$

$$\boxed{B = \frac{-V_0 - H(\gamma-\Gamma)}{2\Gamma}}$$

c) $y(T) = 0$

$$0 = \overbrace{e^{-\gamma T}}^{>0}\left(Ae^{\Gamma T} + Be^{-\Gamma T}\right)$$

$$0 = Ae^{\Gamma T} + Be^{-\Gamma T}$$

$$0 = Ae^{2\Gamma T} + B$$

$$-\frac{B}{A} = e^{2\Gamma T}$$

$$\ln(-B/A) = 2\Gamma T$$

cruzará $y=0$ en el instante $T = \frac{1}{2\Gamma}\ln\left(-\frac{B}{A}\right)$ ssi

$$T>0 \Rightarrow \ln\left(-\frac{B}{A}\right) > 0$$

$$-\frac{B}{A} > 1$$

- $A>0 \longrightarrow B < -A$

$$-V_0 - H(\gamma-\Gamma) < -V_0 - H(\gamma+\Gamma)$$

$H\Gamma < -H\Gamma \longrightarrow$ no es posible $(H, \Gamma > 0)$

- $A<0 \longrightarrow B > -A$

$$-V_0 - H(\gamma-\Gamma) > -V_0 - H(\gamma+\Gamma)$$

$H\Gamma > -H\Gamma \longrightarrow$ es posible

$$A<0 \Rightarrow \frac{V_0 + H(\gamma+\Gamma)}{2\Gamma} < 0 \rightarrow \boxed{V_0 < -H(\gamma+\Gamma)}$$