II) Sumatorias generales

La idea es hacer sumatorias en órdenes distintos, sobre índices distintos, etc.

Ej.: $a_n + a_{n-1} + a_{n-2} + \cdots + a_1$

$$\sum_{k=1}^{n} a_{n+1-k} = \underset{\substack{\uparrow \\ k=1}}{a_{n+1-1}} + \underset{\substack{\uparrow \\ k=2}}{a_{n+1-2}} + \underset{\substack{\uparrow \\ k=3}}{a_{n+1-3}} + \cdots + \underset{\substack{\uparrow \\ k=n}}{a_{n+1-n}}$$

$$= \underset{\substack{\uparrow \\ k=1}}{a_n} + \underset{\substack{\uparrow \\ k=2}}{a_{n-1}} + \underset{\substack{\uparrow \\ k=3}}{a_{n-2}} + \cdots + \underset{\substack{\uparrow \\ k=n}}{a_1}$$

$$= a_1 + a_2 + a_3 + \cdots + a_n$$

$$= \sum_{k=1}^{n} a_k$$

Ej: $\sum_{\substack{k=1 \\ k \text{ par}}}^{2n} k = 2+4+6+\cdots+2n$

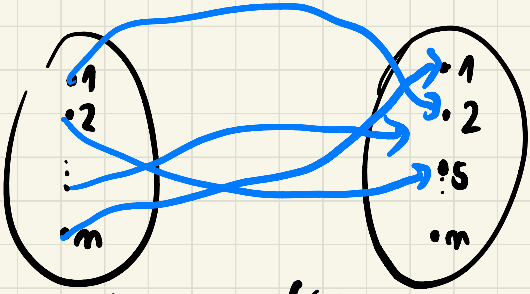
Notación: $[1..n] = \{1,2,3,4,\ldots,n\}$

Prop: Para todo $n \in \mathbb{N}$, para toda secuencia $(a_k)_{k\in[1..n]}$ y para toda $f:[1..n]\to[1..n]$ biyectiva, si $(b_k)_{k\in[1..n]}$ está dada por $b_k = a_{f(k)}$ entonces

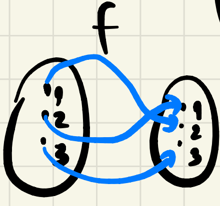
$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} b_k$$

En resumen e intuitivamente: el orden en que sumo no importa mientras sume los mismos términos.

Explicación:

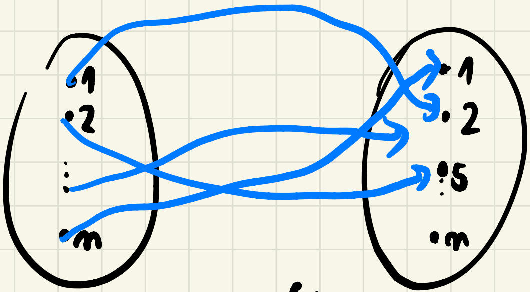


Ej:

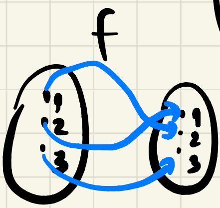


$f(1)=2$ $f(3)=3$
$f(2)=1$

Explicación:



Ej:



$f(1)=2 \quad f(3)=3$

$f(2)=1$

$$\sum_{k=1}^{3} a_k = a_1 + a_2 + a_3$$

$$b_k = a_{f(k)}$$

$$\sum_{k=1}^{3} b_k = \sum_{k=1}^{3} a_{f(k)} = a_{f(1)} + a_{f(2)} + a_{f(3)} = a_2 + a_1 + a_3$$

Intuitivamente, el orden no importa.

Ej: $$\sum_{k=1}^{n} (n-k)^2 = (n-1)^2 + (n-2)^2 + \dots + (n-n)^2$$

$$= \sum_{k=0}^{n-1} k^2 = \frac{(n-1)\,n\,(2(n-1)+1)}{6}$$

Sumatoria sobre un conjunto de índices:

Sea $a:\Omega\to\mathbb{R}$ una función, $n\in\mathbb{N}$ y $f:[1..n]\to\Omega$ biyectiva. Para $b_k = a_{f(k)}$ definimos

$$\sum_{w\in\Omega} a_w = \sum_{k=1}^{n} b_k$$

Si quiero que mis índices sean cosas distintas que los naturales del m a n, uso esto.

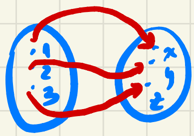
Ej: Si $\Omega = \{x, y, z\}$

$a_x = 3$

$a_y = 5$

$a_z = 12$

$$\sum_{w\in\Omega} a_w = a_x + a_y + a_z$$



$$= a_y + a_z + a_x$$

$$= a_z + a_x + a_y$$

$$= 20$$

Ej: $\displaystyle\sum_{k\in[1..n]} k = 1+2+3+\dots+n = \sum_{k=1}^{n} k$

$$\sum_{\substack{k\in[1..2n] \\ k \text{ par}}} k = 2+4+6+\dots+2n$$

$$\Updownarrow$$

$$\{k\in[1..2n] \mid k \text{ es par}\}$$

Prop: Si $(a_k)_{k\in[1..n]}$ es una secuencia y $I, J \subseteq [1..n]$ son disjuntos $(I\cap J=\emptyset)$, entonces

$$\sum_{k\in I\cup J} a_k = \sum_{k\in I} a_k + \sum_{k\in J} a_k$$

Ej:

k	1	2	3	4	5	6
a	a_1	a_2	a_3	a_4	a_5	a_6

$I = \{1, 5, 6\}$

$J = \{2, 4\}$

$I \cup J = \{1, 2, 4, 5, 6\}$

$I \cap J = \emptyset$

$$\sum_{k \in I \cup J} a_k = a_1 + a_2 + a_4 + a_5 + a_6$$

$$\sum_{k \in I} a_k + \sum_{k \in J} a_k = (a_1 + a_5 + a_6) + (a_4 + a_2)$$

Ej: $$\underbrace{\sum_{k \in [1..n]} a_k}_{I \cup J} = \underbrace{\sum_{\substack{k \in [1..n] \\ k \text{ par}}} a_k}_{I} + \underbrace{\sum_{\substack{k \in [1..n] \\ k \text{ impar}}} a_k}_{J}$$

Ej: Veamos que

$$\sum_{k=0}^{2n} (-1)^k k^2 = n(2n+1)$$

$$(-1)^k = \begin{cases} 1 & \text{si } k \text{ es par} \\ -1 & \text{si } k \text{ es impar} \end{cases}$$

$$\sum_{k=0}^{2n} (-1)^k k^2 = \sum_{\substack{k\in[0..2n] \\ k \text{ par}}} (-1)^k k^2 + \sum_{\substack{k\in[0..2n] \\ k \text{ impar}}} (-1)^k k^2$$

$$= \sum_{\substack{k\in[0..2n] \\ k \text{ par}}} k^2 - \sum_{\substack{k\in[0..2n] \\ k \text{ impar}}} k^2$$

$$\sum_{\substack{k\in[0..2n] \\ k \text{ par}}} k^2 = \sum_{k=0}^{n} (2k)^2 = \sum_{k=0}^{n} 4k^2 = 4\sum_{k=0}^{n} k^2$$

$$= \frac{n(n+1)(2n+1)}{6} \cdot 4$$

$$\sum_{\substack{k\in[0..2n]\\ k \text{ impar}}} k^2 = \sum_{k=0}^{n-1} (2k+1)^2 = \sum_{k=0}^{n-1} (4k^2+4k+1)$$

$$= 4\sum_{k=0}^{n-1} k^2 + 4\sum_{k=0}^{n-1} k + \sum_{k=0}^{n-1} 1$$

$$= 4\frac{(n-1)n(2(n-1)+1)}{6} + 4\frac{(n-1)n}{2} + n$$

Juntando todo:

$$\sum_{k=0}^{2n} (-1)^k k^2 = \sum_{\substack{k\in[0..2n]\\ k \text{ par}}} (-1)^k k^2 + \sum_{\substack{k\in[0..2n]\\ k \text{ impar}}} (-1)^k k^2$$

$$= \sum_{\substack{k\in[0..2n]\\ k \text{ par}}} k^2 - \sum_{\substack{k\in[0..2n]\\ k \text{ impar}}} k^2$$

$$= \frac{2}{3} n(n+1)(2n+1) - \left(4\frac{(n-1)n(2(n-1)+1)}{6} + 4\frac{(n-1)n}{2} + n\right)$$

$$= n(2n+1)$$

Sumatorias dobles

¿Qué pasa si multiplicamos sumatorias?

$$(a_1 + a_2 + \dots + a_n)(b_1 + \dots + b_m)$$

$$= (a_1 b_1 + a_1 b_2 + \dots + a_1 b_m)$$

$$+ (a_2 b_1 + a_2 b_2 + \dots + a_2 b_m)$$

$$+ (a_3 b_1 + a_3 b_2 + \dots + a_3 b_m)$$

$$\vdots$$

$$+ (a_m b_1 + a_m b_2 + \dots + a_m b_m)$$

$$= \sum_{j=1}^{m} a_1 b_j + \sum_{j=1}^{m} a_2 b_j + \dots + \sum_{j=1}^{m} a_n b_j$$

$$= \sum_{k=1}^{n} \underbrace{\left(\sum_{j=1}^{m} a_k b_j \right)}_{c_k} = \left(\sum_{k=1}^{n} a_k \right) \left(\sum_{j=1}^{m} b_j \right)$$

$$= \sum_{k=1}^{n} c_k$$

Ej: $\sum_{k=1}^{n}\left(\sum_{j=1}^{m} k^2 j\right) = \left(\sum_{k=1}^{n} k^2\right)\left(\sum_{j=1}^{m} j\right)$

$a_k = k^2$ $\quad b_j = j$

$$= \left(\frac{n(n+1)(2n+1)}{6}\right)\left(\frac{m(m+1)}{2}\right)$$

Intercambio de sumatorias

Si consideramos que la secuencia depende de k y de j (o sea es $a_{k,j}$) podemos considerar $\sum_{k=1}^{n}\sum_{j=1}^{m} a_{kj}$. ¿Es cierto que

$$\sum_{k=1}^{n}\sum_{j=1}^{m} a_{kj} = \sum_{j=1}^{m}\sum_{k=1}^{n} a_{kj} \,?$$ Sí.

Pensamos este tipo de sumatorias como "recorrer una tabla e ir sumando"

$k \backslash j$	1	2	...	m
1	$a_{1,1}$	$a_{1,2}$	...	$a_{1,m}$
2	$a_{2,1}$	$a_{2,2}$	...	$a_{2,m}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
n	$a_{m,1}$	$a_{m,2}$	...	$a_{m,m}$

$$\sum_{k=1}^{n}\sum_{j=1}^{m} a_{k,j}$$

$$= \sum_{k=1}^{n}(a_{k,1}+a_{k,2}+\dots+a_{k,m})$$

$$= (a_{1,1}+\dots+a_{1,m})$$
$$+ (a_{2,1}+\dots+a_{2,m})$$
$$\vdots$$
$$+ (a_{m,1}+\dots+a_{m,m})$$

$$\sum_{j=1}^{m}\sum_{k=1}^{n} a_{k,j} = \sum_{j=1}^{m}(a_{1,j}+a_{2,j}+a_{3,j}+\dots+a_{n,j})$$

$$= (a_{1,1}+a_{2,1}+\dots+a_{m,1})$$
$$+ (a_{1,2}+a_{2,2}+\dots+a_{m,2})$$
$$\vdots$$
$$+ (a_{1,m}+a_{2,m}+\dots+a_{m,m})$$

Dar vuelta la sumatoria es pasar de sumar filas y luego columnas a columnas y luego filas.

En general, el rango de los índices puede depender de otros índices

Ej: $\sum_{k=1}^{n} \sum_{j=1}^{k} a_{k,j}$

$$\sum_{k=1}^{n} \sum_{j=1}^{k} a_{k,j} = \sum_{k=1}^{m} (a_{k,1} + a_{k,2} + \dots + a_{k,k})$$

$$= (a_{1,1}) + (a_{2,1} + a_{2,2}) + (a_{3,1} + a_{3,2} + a_{3,3})$$
$$+ \dots + (a_{m,1} + a_{m,2} + \dots + a_{m,n})$$

$k \backslash j$	1	2	$\dots$	n
1	$a_{1,1}$	$\cancel{a_{1,2}}$	$\cancel{\dots}$	$\cancel{a_{1,m}}$
2	$a_{2,1}$	$a_{2,2}$	$\cancel{\dots}$	$\cancel{a_{2,m}}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\cancel{\vdots}$
n	$a_{m,1}$	$a_{m,2}$	$\dots$	$a_{m,n}$

De la tabla,

$$\sum_{k=1}^{n} \sum_{j=1}^{k} a_{k,j} = \sum_{j=1}^{m} \sum_{k=j}^{n} a_{k,j}$$