P1 a) $F = (2yz^2, xz^2, 3xyz)$.

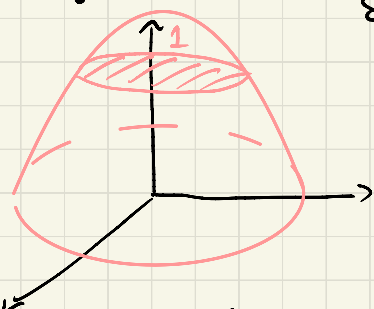
Definamos

$$C = \{x^2+y^2+z^2 = 4\} \cap \{x^2+y^2 = 3z\}.$$

Debemos calcular

$$\int_C F \cdot d\vec{r}.$$

Dibujamos C. Despejando vemos que z satisface



$$(z-1)(z+4) = 0$$

$$\Rightarrow z = 1$$

$\Rightarrow$ $C = \{x^2+y^2 = 3, z = 1\}$ y definimos

$$S = \{x^2+y^2 \leq 3, z = 1\}$$ de forma que

$$C = \partial S.$$

$\Rightarrow$ Stokes:

$$\int_{\partial S} F \cdot d\vec{r} = \int_S (\nabla \times F) \cdot d\vec{A} = I$$

Es decir, basta calcular el flujo.

- Def del flujo

$$\iint_D (\nabla \times F)(\vec{r}(u,v)) \cdot \left[\partial_u \vec{r}(u,v) \times \partial_v \vec{r}(u,v)\right] du\, dv.$$

- Parametrización

$$r(\rho,\theta) = (\rho\cos\theta, \rho\,\text{sen}\,\theta, 1)$$

$$\rho \in [0, \sqrt{3}] \quad , \quad \theta \in [0, 2\pi]$$

- Rotor.

$$(\nabla \times F) = z(x\hat{\imath} + y\hat{\jmath}) - z^2\hat{k}$$

$$= z\hat{\rho} - z^2\hat{k}.$$

- Rotor en la param.

$$(\nabla \times F)(r(\rho,\theta)) = \rho\cos\theta\,\hat{\imath} + \rho\,\text{sen}\,\theta\,\hat{\jmath} - \hat{k}$$

- Normal

$$\partial_\rho \vec{r}(\rho,\theta) \times \partial_\theta \vec{r}(\rho,\theta) = \rho\hat{k}$$

(queda hacia arriba ↑)

Se obtiene

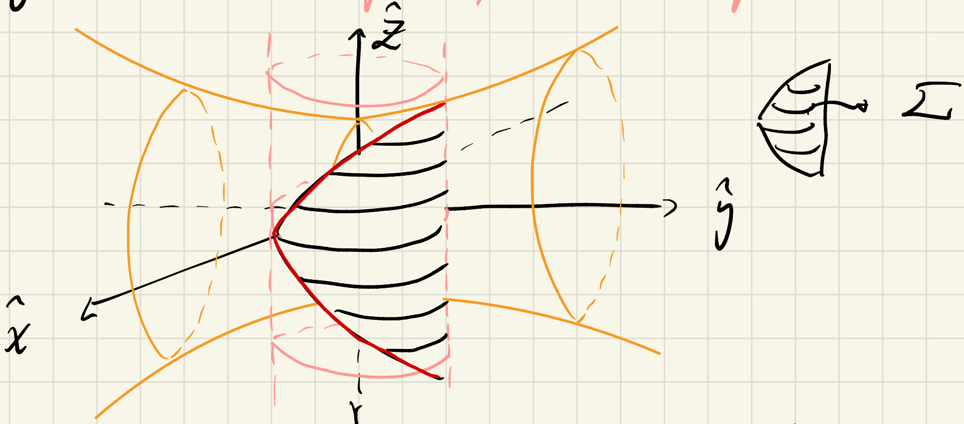
$$\boxed{\int_{\partial S} F \cdot d\vec{r} = -3\pi}$$

b) $S = \{ z^2 + x^2 = 1 + y^2 ,\ y \geq 0 \}$

$C = S \cap \{ x^2 + y^2 = 1,\ z \in \mathbb{R} \}$

$¿ \oint_C F \cdot d\vec{r} \ ?$

Dibujamos (está difícil, haré mi mejor intento)



Acá, C es la sección en rojo (que se cierra en el otro lado de la figura)

Definamos Σ como la superficie t.q.

$$\partial \Sigma = C$$

$\Rightarrow$ Stokes

$$\int_{\partial \Sigma} F \cdot d\vec{r} = \int_{\Sigma} (\nabla \times F) \cdot d\vec{A}$$

- Def del flujo

$$\iint_D (\nabla \times F)(\vec{r}(u,v)) \cdot [\partial_u \vec{r} \times \partial_v \vec{r}]\, du\, dv$$

- Parametrización

$z = \pm\sqrt{2}\, y$

$$r(\theta, z) = (\cos\theta,\ \sin\theta,\ z)$$

$$\theta \in [0, \pi], \quad z \in [-\sqrt{2}\sin\theta, \sqrt{2}\sin\theta]$$

- $\nabla \times F$ en cilíndricos:

$$\frac{1}{h_\rho h_\theta h_z}\begin{vmatrix} h_\rho \hat{\rho} & h_\theta \hat{\theta} & h_z \hat{z} \\ \partial_\rho & \partial_\theta & \partial_z \\ \rho^2 e^{z} \sin^4\theta & \rho\left(\frac{z}{\rho} + \theta\right) & \sin\theta + \sqrt{1+\rho^2} \end{vmatrix}$$

$$= \frac{1}{\rho}\left(\hat{\rho}(\cos\theta - 1) \underbrace{- \cdots}_{\text{El resto de las coord. no son necesarias}} \right)$$

- Rotor en param.

$$(\nabla \times F)(\vec{r}(\theta, z)) = \hat{\rho}(\cos\theta - 1)$$

- Normal

$$\partial_\theta \vec{r} \times \partial_z \vec{r} = \hat{\rho}$$

Finalmente

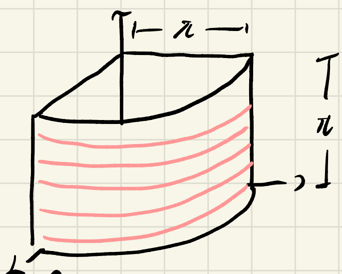
$$\int_0^{\pi}\int_{-\sqrt{2}\sin\theta}^{\sqrt{2}\sin\theta} (\cos\theta - 1)\,\hat{\rho}\cdot\hat{\rho}\;dz\,d\theta$$

$$= 2\sqrt{2}\left(\int_0^{\pi} \cos\theta\sin\theta\,d\theta - \int_0^{\pi}\sin\theta\,d\theta\right)$$

$$= -4\sqrt{2}$$

$$\Rightarrow \quad \boxed{\oint_C F\cdot d\vec{r} = -4\sqrt{2}}$$

P2 Cerramos el volumen y usamos divergencia



$$\int_{\partial\Omega} F\cdot d\vec{A} = \int_{\Omega} (\nabla\cdot F)\,dV$$

Ω corresponde al cuarto de cilindro y queda parametrizado por

$$\vec{r}(\rho,\theta,z) = (\rho\cos\theta, \rho\sin\theta, z)$$

$$z\in[0,\pi],\quad \rho\in[0,\pi],\quad \theta\in[0,\pi/2]$$

La divergencia

$$\nabla \cdot F = y^2 z + x^2 z + 0 = \rho^2 z$$

El dV

$$|(\partial_\rho \vec{r} \times \partial_\theta \vec{r}) \cdot \partial_z \vec{r}| \, d\rho \, d\theta \, dz$$

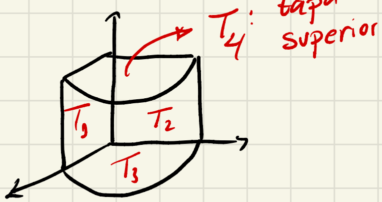
$$= |(h_\rho \hat{\rho} \times h_\theta \hat{\theta}) \cdot h_z \hat{z}| \, d\rho \, d\theta \, dz = \rho \, d\rho \, d\theta \, dz$$

$$\Rightarrow \int_\Omega \nabla \cdot F \, dV = \int_0^{\pi/2} \int_0^{\pi} \int_0^{\pi} \rho^2 z \, \rho \, d\rho \, d\theta \, dz$$

$$= \frac{\pi^2}{16}.$$

Por otro lado:

$$\int_{\partial\Omega} F \cdot d\vec{A} = \underbrace{\int_M F \cdot d\vec{A}}_{\text{Esto queremos calcular!}} + \underbrace{\sum_{i=1}^{4} \int_{T_i} F \cdot dA}$$



T_3 $\quad \vec{r}(\rho,\theta) = (\rho\cos\theta, \rho \,\text{sen}\theta, 0)$

$\rho \in [0,\pi]$, $\theta \in [0, \pi/2]$

$$\partial_\rho \vec{r} \times \partial_\theta \vec{r} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos\theta & \text{sen}\theta & 0 \\ -\rho\,\text{sen}\theta & \rho\cos\theta & 0 \end{vmatrix}$$

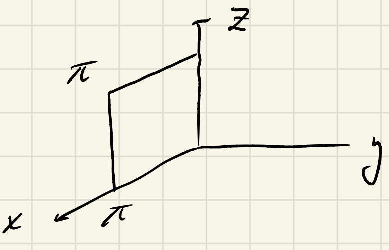
$$= \rho \hat{k} \quad \hat{\mathrm{I}}$$

$$\Rightarrow \int_{T_3} \underbrace{\begin{pmatrix} \cancel{\cdot} \\ \cancel{\cdot} \\ x^2 e^y \end{pmatrix}}_{\text{No depende la altura } z.} \cdot \rho \hat{k} \, d\rho d\theta$$

T_4 Acá es análogo salvo que la normal cambia de signo

$\Rightarrow$ Flujo (T_3) + Flujo $(T_4) = 0$.

T_1



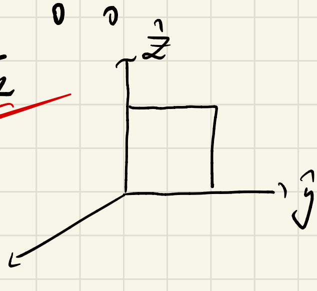
$\vec{r}(x,z) = (x, 0, z)$

$$d\vec{A} = (\partial_x \vec{r} \times \partial_z \vec{r})\, dx dz = -\hat{\jmath}\, dx dz$$

$$\Rightarrow \int_0^\pi \int_0^\pi \begin{pmatrix} \cancel{\cdot} \\ e^x \cos z + x^2 0 z \\ \cancel{\cdot} \end{pmatrix} (-\hat{\jmath})\, dx dz$$

$$= -\int_0^\pi \int_0^\pi e^x \cos z \, dx\, dz = -(e^\pi - 1)\int_0^\pi \cos z \, dz = 0$$

T_2



$$\vec r(y,z) = (0, y, z)$$

$$\Rightarrow d\vec A = \left(\partial_z \vec r \times \partial_y \vec r\right) dz\, dy$$

$$= -\hat\imath \, dz\, dy$$

$$= \int_0^\pi \int_0^\pi \left(e^z \sin y + 0\; y^2 z\right)\cdot(-\hat\imath)\, dz\, dy$$

$$= -\int_0^\pi \int_0^\pi e^z \sin y \, dz\, dy = -(e^\pi - 1)\int_0^\pi \sin y \, dy$$

$$= -(e^\pi - 1)\, 2$$

$$\Rightarrow \boxed{\int_\Gamma F \cdot d\vec A = \frac{\pi^7}{16} + 2(e^\pi - 1)}$$

P3

(a)

Cilíndricas.

$$\vec{r} : [0, +\infty) \times [0, 2\pi) \times \mathbb{R}$$
$$(\rho, \theta, z) \longmapsto (\rho\cos\theta, \rho \operatorname{sen}\theta, z)$$

$$\hat{\rho} = \frac{\partial r}{\partial \rho} \Big/ \left\| \frac{\partial r}{\partial \rho} \right\| = \cos\theta \, \hat{i} + \operatorname{sen}\theta \, \hat{j}$$

$$\hat{\theta} = \frac{\partial r}{\partial \theta} \Big/ \left\| \frac{\partial r}{\partial \theta} \right\| = -\operatorname{sen}\theta \, \hat{i} + \cos\theta \, \hat{j}$$

$$\hat{z} = \frac{\partial r}{\partial z} \Big/ \left\| \frac{\partial r}{\partial z} \right\| = \hat{k}$$

$$\Rightarrow F(\rho, \theta, z) =$$

$$= \frac{1}{\rho^2} \Big[\left(\rho\cos\theta - \rho \operatorname{sen}\theta \, \rho \arctan(z^2) \right) \hat{i}$$
$$+ \left(\rho \operatorname{sen}\theta + \rho\cos\theta \, \rho \arctan(z^2) \right) \hat{j}$$
$$+ z\rho^2 \hat{k} \Big]$$

$$= \frac{1}{\rho} \Big(\cos\theta \, \hat{i} + \operatorname{sen}\theta \, \hat{j} \Big)$$
$$+ \arctan(z^2) \left(-\operatorname{sen}\theta \, \hat{i} + \cos\theta \, \hat{j} \right)$$
$$+ z \hat{k}$$

$$= \frac{1}{\rho}\hat{\rho} + \arctan(z^2)\,\hat{\theta} + z\,\hat{k}.$$

queda bien definido cuando estamos en una bola $B(x_0, R)$ que intersecta el eje z

(b) Divergencia en cilíndricas

$$\nabla\cdot F = \frac{1}{\rho}\left[\partial_\rho\left(\frac{1}{\rho}\cdot\rho\right) + \partial_\theta\left(\arctan(z^2)\right) + \partial_z\left(z\cdot\rho\right)\right]$$

$$= 1$$

(c) Tomamos una bola $B(x_0, R)$ que no intersecte el eje z. Por Gauss:

$$\int_{\partial B(x_0,R)} F\cdot d\vec{A} = \int_{B(x_0,R)} \nabla\cdot F \; dV$$

el teo es con normal interior.

$$= \text{Volumen de } B(x_0,R)$$

$$= \frac{4}{3}\pi R^3$$

Pero como queremos con normal exterior

$$\Rightarrow \; -\frac{4}{3}\pi R^3.$$

El P4 queda propuesto

MATH