Aux 10 PyE

P1) a) $X \sim \text{Berno}(p)$

$$\Rightarrow \Phi_X(t) = \mathbb{E}(e^{-tX}) = e^{-t}\,\mathbb{P}(X=1) + e^{0}\,\mathbb{P}(X=0)$$
$$= e^{-t}\cdot p + 1 - p$$
$$= 1 - p(1-e^{-t})$$

b) $Y \sim \text{Binom}(n,p)$

$$\Phi_Y(t) = \mathbb{E}(e^{-tY}) = \sum_{k=0}^{n} e^{-tk}\,\mathbb{P}(Y=k) = \sum_{k=0}^{n} e^{-tk}\binom{n}{k}p^k(1-p)^{n-k}$$
$$= \sum_{k=0}^{n}\binom{n}{k}(e^{-t}p)^k(1-p)^{n-k} = (1-p+e^{-t}p)^n$$
$$= \left[1-p(1-e^{-t})\right]^n$$

c) Notamos $\Phi_Y(t) = (\Phi_X(t))^n = \prod_{i=1}^{n}\Phi_X(t)$

Esto nos dice que $Y = \sum_{i=1}^{n} X_i$, con $X_i \sim \text{Berno}(p)$; X_i indep

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P2) a) $X \sim \text{Gamma}(\alpha,\beta) \Rightarrow f_X(x) = \dfrac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$

$$\Rightarrow \Phi_X(s) = \mathbb{E}(e^{-sX}) = \int_0^{+\infty} e^{-sx}\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}\,dx = \int_0^{+\infty}\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-(\beta+s)x}\,dx$$

$$= \frac{\beta^\alpha}{(\beta+s)^\alpha}\underbrace{\int_0^{+\infty}\frac{(\beta+s)^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-(\beta+s)x}\,dx}_{=1} = \left(\frac{\beta+s}{\beta}\right)^{-\alpha} = \left(1+\frac{s}{\beta}\right)^{-\alpha}$$

*por ser $\int_0^{+\infty} f_Y(y)\,dy$

con $Y \sim \text{Gamma}(\alpha, \beta+s)$

P2) b) Por prop., def $Y = \sum_{k=1}^{n} X_k$

$$\Rightarrow \Phi_Y(s) = \prod_{i=1}^{n} \Phi_{X_i}(s) = \prod_{i=1}^{n} \left(\frac{s}{\beta}+1\right)^{-\alpha_i} = \left(\frac{s}{\beta}+1\right)^{-\sum \alpha_i}$$

$$\therefore Y = \sum_{i=1}^{n} X_i \sim \text{Gamma}\left(\sum_{i=1}^{n} \alpha_i, \beta\right) \blacksquare$$

P3) a) $X \sim \text{Poiss}(\lambda)$

$$\Rightarrow \Phi_X(s) = \mathbb{E}\left(e^{-sX}\right) = \sum_{k\geq 0} e^{-sk} e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= \sum_{k\geq 0} e^{-\lambda} \frac{(e^{-s}\lambda)^k}{k!} = \frac{e^{-\lambda}}{e^{-\lambda e^{-s}}} \underbrace{\sum_{k\geq 0} e^{-\lambda e^{-s}} \frac{(e^{-s}\lambda)^k}{k!}}_{=1}$$

$$= e^{-\lambda + \lambda e^{-s}}$$

$$= e^{-\lambda(1-e^{-s})} \quad \#$$

*por ser $\sum_{k\geq 0} \mathbb{P}(Y=k)$, con $Y \sim \text{Poiss}(e^{-s}\lambda)$

b) Por prop.:

$$\Phi_Y(s) = \prod_{i=1}^{n} \Phi_{X_i}(s) = \prod_{i=1}^{n} e^{-\lambda_i(1-e^{-s})} = e^{-\sum_{i=1}^{n}[\lambda_i(1-e^{-s})]} = e^{-(1-e^{-s})\sum_{i=1}^{n}\lambda_i}$$

$$\therefore Y \sim \text{Poisson}(\eta), \text{ con } \eta = \sum_{i=1}^{n} \lambda_i \blacksquare$$

P4) Notemos que $X^s = X - Y = X + (-Y)$

luego $\Phi_{X^s}(t) = \Phi_X(t) \cdot \Phi_{-Y}(t) = \Phi_X(t) \cdot \overline{\Phi_Y(t)}$

- Pero como $F_Y = F_X \Rightarrow \Phi_Y = \Phi_X$, luego:

$$\Phi_{X^s}(t) = \Phi_X(t) \cdot \overline{\Phi_X(t)} = |\Phi_X(t)|^2 \blacksquare$$