Pauta C2 PyE

P1) a) Notemos que por enunciado, se inicia con capital C

Cuando se gana una partida, se duplica el capital

$\Rightarrow$ Si se ganan k partidas, el capital se multiplica por 2^k

Cuando se pierde, se divide a la mitad el capital

$\Rightarrow$ Si se pierden l partidas, se multiplica por $1/2^l$ el capital

$\Rightarrow Z = C \frac{2^k}{2^l} = C2^{k-l}$, si defino X la v.a. para partidas ganadas

Y la v.a. para partidas perdidas

$$\Rightarrow Z = C2^{X-Y} \quad \blacksquare$$

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b) Notemos que $Z = C2^{X-Y} = C2^{X-(n-X)} = C2^{2X-n}$.

Sea $k \in \{0 \dots n\}$:

$$\mathbb{P}(\log_2 Z = \log_2 C - n + 2k) = \mathbb{P}(\log_2(C2^{2X-n}) = \log_2 C - n + 2k)$$

$$= \mathbb{P}(\log_2(C) + \log_2(2^{2X-n}) = \log_2 C - n + 2k)$$

$$= \mathbb{P}((2X-n)\log_2(2) = -n + 2k)$$

$$= \mathbb{P}(2X - n = 2k - n) = \mathbb{P}(X = k)$$

Como X son partidas ganadas, con prob p se gana, y hay n juegos

$$\Rightarrow X \sim \text{Bin}(n, p)$$

$$\therefore \mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\Rightarrow \mathbb{P}(\log_2 Z = \log_2 C - n + 2k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \blacksquare$$

P2) a) • Notemos que $b>0$, $\mu \in \mathbb{R}$, luego

$$f_X(x) = \frac{1}{2b} e^{\frac{-|x-\mu|}{b}} \geq 0 \quad \forall x \in \mathbb{R}$$

• $$\int_{-\infty}^{+\infty} f_X(x)\,dx = \int_{-\infty}^{+\infty} \frac{1}{2b} e^{\frac{-|x-\mu|}{b}}\,dx = \frac{1}{2b}\int_{-\infty}^{+\infty} e^{\frac{-|x-\mu|}{b}}\,dx = \frac{1}{2b}\left(\int_{-\infty}^{\mu} e^{\frac{-|x-\mu|}{b}}\,dx + \int_{\mu}^{+\infty} e^{\frac{-|x-\mu|}{b}}\,dx\right)$$

$$= \frac{1}{2b}\left(\int_{-\infty}^{\mu} e^{\frac{-(\mu-x)}{b}}\,dx + \int_{\mu}^{+\infty} e^{\frac{-(x-\mu)}{b}}\,dx\right) = \frac{1}{2b}\left(\int_{-\infty}^{\mu} e^{\frac{x-\mu}{b}}\,dx + \int_{\mu}^{+\infty} e^{\frac{\mu-x}{b}}\,dx\right)$$

$$= \frac{1}{2b}\left(b e^{\frac{x-\mu}{b}}\Big|_{-\infty}^{\mu} + -b e^{\frac{\mu-x}{b}}\Big|_{\mu}^{+\infty}\right) = \frac{1}{2}\left(e^{\frac{x-\mu}{b}}\Big|_{-\infty}^{\mu} - e^{\frac{\mu-x}{b}}\Big|_{\mu}^{+\infty}\right)$$

$$= \frac{1}{2}\left(1 - 0 - (0-1)\right) = \frac{1}{2}\cdot 2 = 1 \quad \blacksquare$$

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b) $$\mathbb{E}(X) = \int_{-\infty}^{+\infty} x\, f_X(x)\,dx = \int_{-\infty}^{+\infty} x \frac{1}{2b} e^{\frac{-|x-\mu|}{b}}\,dx = \frac{1}{2b}\int_{-\infty}^{+\infty} x\, e^{-\left|\frac{x-\mu}{b}\right|}\,dx$$

Def $z = \frac{x-\mu}{b} \Rightarrow x = bz+\mu \Rightarrow dx = b\,dz$

$$\Rightarrow \mathbb{E}(X) = \frac{1}{2b}\int_{-\infty}^{+\infty} (bz+\mu) e^{-|z|}\, b\,dz = \frac{1}{2}\int_{-\infty}^{+\infty} (bz+\mu) e^{-|z|}\,dz$$

$$= \frac{1}{2}\left[b\int_{-\infty}^{+\infty} z e^{-|z|}\,dz + \mu\int_{-\infty}^{+\infty} e^{-|z|}\,dz\right] = \frac{1}{2}\left[b\left(\int_{-\infty}^{0} z e^{-|z|}\,dz + \int_{0}^{+\infty} z e^{-|z|}\,dz\right) + \mu\int_{-\infty}^{+\infty} e^{-|z|}\,dz\right]$$

$$= \frac{1}{2}\left[b\left(\int_{-\infty}^{0} z e^{z}\,dz + \int_{0}^{+\infty} z e^{-z}\,dz\right) + \mu\left(\int_{-\infty}^{0} e^{z}\,dz + \int_{0}^{+\infty} e^{-z}\,dz\right)\right]$$

$$= \frac{1}{2}\left[b\underbrace{\left(\int_{0}^{+\infty} (-z) e^{-z}\,dz + \int_{0}^{+\infty} z e^{-z}\,dz\right)}_{=0} + \mu\left(\int_{0}^{+\infty} e^{-z}\,dz + \int_{0}^{+\infty} e^{-z}\,dz\right)\right]$$

$$= \frac{1}{2}\cdot\mu\cdot 2\int_{0}^{+\infty} e^{-z}\,dz = \mu\int_{0}^{+\infty} e^{-z}\,dz = \mu\left(-e^{-z}\right)\Big|_{0}^{+\infty} = \mu(0+1) = \mu \quad \blacksquare$$

P2] c) Supongamos $\mu = 0$, def $Y = |X|$, luego $Y \geq 0$

Por teo. de cambio de variable, sea $y > 0$.

$$f_Y(y) = \sum_{x_y \in h^{-1}(y)} |h'(x_y)|^{-1} f_X(x_y)$$

Notemos $h(x) = |x| \Rightarrow h'(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$

Además si $y = |x| \Rightarrow x = \pm y$

Sea $y > 0$

$$\therefore f_Y(y) = |h'(y)|^{-1} f_X(y) + |h'(-y)|^{-1} f_X(-y)$$

$$= \frac{1}{1} \cdot \frac{1}{2b} e^{-\frac{|y|}{b}} + \frac{1}{1} \cdot \frac{1}{2b} e^{-\frac{|-y|}{b}}$$

$$= \frac{1}{2b} \left(e^{-y/b} + e^{-y/b} \right) = 1/b \, e^{-y/b}$$

$$\Rightarrow f_Y(y) = \begin{cases} \frac{1}{b} e^{-\frac{y}{b}} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

$\blacksquare$

P3) Def $X = \frac{1}{2} Y^2$, veamos que $X \sim$ Gamma $(1/2)$

Como $Y \sim N(0,1) \Rightarrow f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} y^2}$, $y \in \mathbb{R}$

- Como $X = 1/2\, Y^2 \Rightarrow X \geq 0$; sea $x \geq 0$; por Teo. cambio variable:

$$f_X(x) = \sum_{y_x \in h^{-1}(x)} |h'(y_x)|^{-1} f_Y(y_x)$$

↳ Sea $x = h(y) = \frac{1}{2} y^2 \Rightarrow y^2 = 2x \Rightarrow y = \pm \sqrt{2x}$

Ademas $h'(y) = y$

$$\Rightarrow f_X(x) = |h'(\sqrt{2x})|^{-1} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\sqrt{2x})^2} + |h'(-\sqrt{2x})|^{-1} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\cdot(-\sqrt{2x})^2}$$

$$= |\sqrt{2x}|^{-1} \cdot \frac{1}{\sqrt{2\pi}} e^{-x} + |-\sqrt{2x}|^{-1} \cdot \frac{1}{\sqrt{2\pi}} e^{-x}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2x}} e^{-x} + \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2x}} e^{-x}$$

$$= \frac{2}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2}} x^{-1/2} e^{-x}$$

$$= \frac{2\Gamma(1/2)}{\sqrt{2\pi}} \cdot x^{1/2 - 1} e^{-x}$$

$$= C x^{1/2-1} e^{-x}$$

$$\therefore f_X(x) = \begin{cases} C x^{1/2-1} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases} \Rightarrow X \sim \text{Gamma}(1/2) \quad \boxtimes$$

P4) a) Sea $K \in \mathbb{N}$:

$$\bullet\ \mathbb{P}(X=K) = \mathbb{P}(YZ=K) = \mathbb{P}(YZ=K \mid Z=1)\,\mathbb{P}(Z=1) + \mathbb{P}(YZ=K \mid Z=0)\,\mathbb{P}(Z=0)$$

$$\overset{\text{indep}}{=} \mathbb{P}(Y=K)\,\mathbb{P}(Z=1) + \mathbb{P}(0=K)\,\mathbb{P}(Z=0)$$

$$= \frac{e^{-\lambda}\lambda^K}{K!}\cdot p + (1-p)\,\mathbb{1}_{\{0\}}(K) \quad \blacksquare$$

P4) b)

$$\mathbb{E}(X) = \sum_{K=0}^{\infty} K\,\mathbb{P}(X=K) = \sum_{K=1}^{\infty} K\,\mathbb{P}(X=K)$$

$$= \sum_{K=1}^{\infty}\left[\frac{e^{-\lambda}\lambda^K}{K!}p + (1-p)\,\mathbb{1}_{\{0\}}(K)\right]K = \sum_{K=1}^{\infty} K\frac{e^{-\lambda}\lambda^K}{K!}\cdot p$$

$$= p\sum_{K=1}^{\infty}\frac{e^{-\lambda}\lambda^K}{(K-1)!} = p\sum_{K=0}^{\infty}\frac{e^{-\lambda}\lambda^{K+1}}{K!} = \lambda p\underbrace{\sum_{K=0}^{\infty}\frac{e^{-\lambda}\lambda^K}{K!}}_{=1}$$

$$\therefore\ \mathbb{E}(X) = \lambda p \quad \blacksquare$$

* Otra forma de hacerlo es por independencia:

$$\mathbb{E}(X) = \mathbb{E}(YZ) \overset{\text{indep}}{=} \mathbb{E}(Y)\,\mathbb{E}(Z) = \lambda p \quad \blacksquare$$