P3) a) (Ex PyE

- $\mathbb{E}(e^{-sY}) = \sum_{n\geq 1} e^{-sn}\,\mathbb{P}(Y=n) = \sum_{n\geq 1} e^{-sn} p(1-p)^{n-1}$

$$= e^{-s}\sum_{n\geq 1} e^{-s(n-1)} p(1-p)^{n-1} = e^{-s}\sum_{n\geq 1} p\left[e^{-s}(1-p)\right]^{n-1}$$

$$= \frac{e^{-s}p}{1-e^{-s}(1-p)}\sum_{n\geq 1}\left(1-e^{-s}(1-p)\right)\left(e^{-s}(1-p)\right)^{n-1}$$

$$= \frac{p}{e^{s}(1-e^{-s}(1-p))}\underbrace{\sum_{n\geq 1}\left(1-e^{-s}(1-p)\right)\left(e^{-s}(1-p)\right)^{n-1}}_{=\sum_{n\geq 1}\mathbb{P}(Z=n),\ \text{con } Z\sim \text{Geom}(1-e^{-s}(1-p)) \ = 1}$$

$$\Rightarrow \mathbb{E}(e^{-sY}) = \frac{p}{e^{s}-(1-p)} \ ∎$$

- $\mathbb{E}(Y) = -1\cdot \frac{d\psi_Y}{ds}(0) = -1\cdot \frac{-1}{p} = \frac{1}{p}$ ∎

$*\ \psi_Y'(s) = -p\left(e^{s}-(1-p)\right)^{-2}\cdot e^{s} \Rightarrow \psi_Y'(0) = -\frac{1}{p}$

- $\mathbb{E}(Y^2) = \frac{d^2\psi_Y}{ds^2}(0) = \frac{2}{p^2}-\frac{1}{p}$

$*\ \psi_Y''(s) = -p\left(e^{s}\cdot -2\left(e^{s}-(1-p)\right)^{-3}\cdot e^{s} + \left(e^{s}-(1-p)\right)^{-2}e^{s}\right)$

$$\Rightarrow \psi_Y''(0) = -p\left(-2\left(\tfrac{1}{p}\right)^{3} + \left(\tfrac{1}{p}\right)^{2}\right) = \frac{2}{p^2}-\frac{1}{p}$$

$$\Rightarrow \text{Var}(Y) = \mathbb{E}(Y^2)-\mathbb{E}(Y)^2 = \frac{2}{p^2}-\frac{1}{p}-\frac{1}{p^2} = \frac{1}{p^2}-\frac{1}{p} = \frac{1-p}{p^2} \ ∎$$

b) $\psi_X(s) = \prod_{j=1}^{r} \psi_{Y_j}(s) = \left(\dfrac{p}{e^{s}-(1-p)} \right)^{r}$ $\blacksquare$

- $\mathbb{E}(X) = \sum_{j=1}^{r} \mathbb{E}(Y_j) = r/p$ $\blacksquare$

- $\operatorname{Var}(X) \overset{iid}{=} \sum_{j=1}^{r} \operatorname{Var}(Y_j) = \dfrac{r(1-p)}{p^2}$

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c) $\psi_Z(s) = \mathbb{E}(e^{-sZ}) = \sum_{n\geq r} e^{-sn}\, \mathbb{P}(Z=n) = \sum_{n\geq r} e^{-sn} \binom{n-1}{r-1} p^{r}(1-p)^{n-r}$

$$= e^{-sr} \sum_{n\geq r} e^{-s(n-r)} \binom{n-1}{r-1} p^{r} (1-p)^{n-r}$$

$$= e^{-sr} \sum_{n\geq r} \binom{n-1}{r-1} p^{r} \left(e^{-s}(1-p)\right)^{n-r} = \frac{(e^{-s}p)^{r}}{\left(1-e^{-s}(1-p)\right)^{r}} \sum_{n\geq r} \binom{n-1}{r-1} \left(1-e^{-s}(1-p)\right)^{r} \left(e^{-s}(1-p)\right)^{n-r}$$

$$= \left(\frac{p}{e^{s}-(1-p)} \right)^{r} \sum_{n\geq r} \binom{n-1}{r-1} \left(1-e^{-s}(1-p)\right)^{r} \left(e^{-s}(1-p)\right)^{n-r}$$

$$= \left(\frac{p}{e^{s}-(1-p)} \right)^{r} \underbrace{\sum_{n\geq r} \mathbb{P}(T=n)}_{=1}$$, con $T \sim$ Binomial negativa $\left(r, 1-e^{-s}(1-p)\right)$

$\therefore\ \psi_Z(s) = \left(\dfrac{p}{e^{s}-(1-p)} \right)^{r} \Rightarrow \psi_Z(s) = \psi_X(s)$ $\blacksquare$

P3) a) Usando TCV, con $h(u_1,u_2) = (u_1, z) = (u_1, u_1 - u_2)$ inyectiva

- $J_h(u_1,u_2) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} = -1$ $\qquad$ • $h^{-1}(u_1,z) = (u_1, u_1 - z)$

$$\therefore\ f_{u_1,z}(u_1,z) = \left|\det\left(J_h(h^{-1}(u_1,z))\right)\right|^{-1} f_{u_1,u_2}(h^{-1}(u_1,z))$$

$$\overset{\text{indep.}}{=} f_{u_1}(u_1)\cdot f_{u_2}(u_1 - z)$$

$$= \mathbb{1}_{(0,1)}(u_1)\cdot \mathbb{1}_{(0,1)}(u_1 - z) \quad //$$

$$\Rightarrow f_Z(z) = \int_{-\infty}^{+\infty} f_{u_1,z}(u_1,z)\,du_1 = \int_{-\infty}^{+\infty} \mathbb{1}_{(0,1)}(u_1)\,\mathbb{1}_{(0,1)}(u_1 - z)\,du_1 = \int_0^1 \mathbb{1}_{(0,1)}(u_1 - z)\,du_1$$

$$= \int_0^1 \mathbb{1}_{0<u_1-z<1}\,du_1$$

$$= \int_0^1 \mathbb{1}_{z<u_1<1+z}\,du_1 = (1-z)\,\mathbb{1}_{0<z<1} + (1+z)\,\mathbb{1}_{-1<z<0}$$

$$\Rightarrow f_Z(z) = (1-|z|)\,\mathbb{1}_{|z|<1} \quad \blacksquare$$

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b) Usando TCV, con $h(W) = |W| =: Y$ no inyectiva (Def $Y = |W|$)

- $h^{-1}(y) = \{-y, y\}$ $\quad ; \quad h'(w) = \begin{cases} -1 & w<0 \\ 1 & w>0 \end{cases}$

Sea $y>0$

$$\Rightarrow f_Y(y) = |h'(-y)|^{-1} f_W(-y) + |h'(y)|^{-1} f_W(y)$$

$$= f_W(-y) + f_W(y) = (1-|-y|)\,\mathbb{1}_{|-y|<1} + (1-|y|)\,\mathbb{1}_{|y|<1}$$

$$= (1-y)\,\mathbb{1}_{0<y<1} + (1-y)\,\mathbb{1}_{0<y<1} = 2(1-y)\,\mathbb{1}_{0<y<1} \quad \blacksquare$$

c) Def $M = \min(|W|, 1-|W|)$, estudiemos $F_M(m)$, $m \in \mathbb{R}$

- $F_M(m) = 1 - \mathbb{P}(M > m) = 1 - \mathbb{P}(|W| > m, 1-|W| > m)$

$$= 1 - \mathbb{P}(|W| > m, |W| < 1-m)$$

$$= 1 - \mathbb{P}(m < |W| < 1-m)$$

$$= 1 - \int_m^{1-m} f_{|W|}(t)\,dt = 1 - \int_m^{1-m} 2(1-t)\mathbb{1}_{0<t<1}\,dt$$

1) Si $m < 0$:

$$F_M(m) = 1 - \int_0^1 2(1-t)\,dt = 1 - 2\left(\int_0^1 1\,dt - \int_0^1 t\,dt\right)$$

$$= 1 - 2\left(1 - 1/2\right) = 1 - 1 = 0 \,//$$

2) Si $m > 1/2 \Rightarrow 1-m < m$

$$\Rightarrow F_M(m) = 1 \,//$$

3) Si $0 < m < 1/2$:

$$F_M(m) = 1 - \int_m^{1-m} 2(1-t)\,dt = 1 - 2\left(\int_m^{1-m} 1\,dt - \int_m^{1-m} t\,dt\right)$$

$$= 1 - 2\left(1 - 2m - \frac{(1-m)^2}{2} + \frac{m^2}{2}\right)$$

$$= 1 - \left(2 - 4m - (1 - 2m + m^2) + m^2\right)$$

$$= \cancel{1} - 2 + 4m \cancel{+1} - 2m \cancel{+m^2} \cancel{-m^2}$$

$$= 2m$$

$$\therefore F_M(m) = \begin{cases} 1 & m > 1/2 \\ 2m & 0 < m < 1/2 \\ 0 & m < 0 \end{cases} \Rightarrow M \sim \mathrm{Unif}(0, 1/2)$$

P3) a) Sea $y \leq -a$ * Hacer un grafico Y v/s X sirve mucho

- $\mathbb{P}(Y \leq y) = \mathbb{P}(X \mathbb{1}_{|X| \leq a} - X \mathbb{1}_{|X| > a} \leq y)$

$= \mathbb{P}(X > -y) = \mathbb{P}(-X \leq y)$

* Como $X \sim N(0,1)$, su densidad es simétrica c/r al 0

$\Rightarrow -X \sim N(0,1)$

$\therefore \mathbb{P}(Y \leq y) = \mathbb{P}(-X \leq y) = \mathbb{P}(X \leq y)$ //

- Sea $y \geq a$

$\mathbb{P}(Y \leq y) = \mathbb{P}(-y \leq X < -a) + \mathbb{P}(-a \leq X \leq a) + \mathbb{P}(a < X)$

$= \mathbb{P}(-y \leq X) = \mathbb{P}(-X \leq y) = \mathbb{P}(X \leq y)$

- Sea $y \in (-a, a)$:

$\mathbb{P}(Y \leq y) = \mathbb{P}(-a \leq X \leq y) + \mathbb{P}(a < X) = \mathbb{P}(-a \leq X \leq y) + \mathbb{P}(-X < -a)$

$= \mathbb{P}(-a \leq X \leq y) + \mathbb{P}(X < -a)$

$= \mathbb{P}(X \leq y)$

$\therefore \mathbb{P}(Y \leq y) = \mathbb{P}(X \leq y) \quad \forall y \in \mathbb{R} \Rightarrow Y \sim N(0,1)$ ☒

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b) $\text{Cov}(X, Y) = \text{Cov}(X, X \mathbb{1}_{|X| \leq a} - X \mathbb{1}_{|X| > a})$

$= \text{Cov}(X, X \mathbb{1}_{|X| \leq a}) - \text{Cov}(X, X \mathbb{1}_{|X| > a})$

$= \int_{-a}^{a} x^2 f_X(x)\,dx - \left(\int_{-\infty}^{-a} x^2 f_X(x)\,dx + \int_{a}^{+\infty} x^2 f_X(x)\,dx \right)$

$= \int_{-a}^{a} x^2 f_X(x)\,dx - \int_{-\infty}^{-a} x^2 f_X(x)\,dx - \int_{a}^{+\infty} x^2 f_X(x)\,dx$

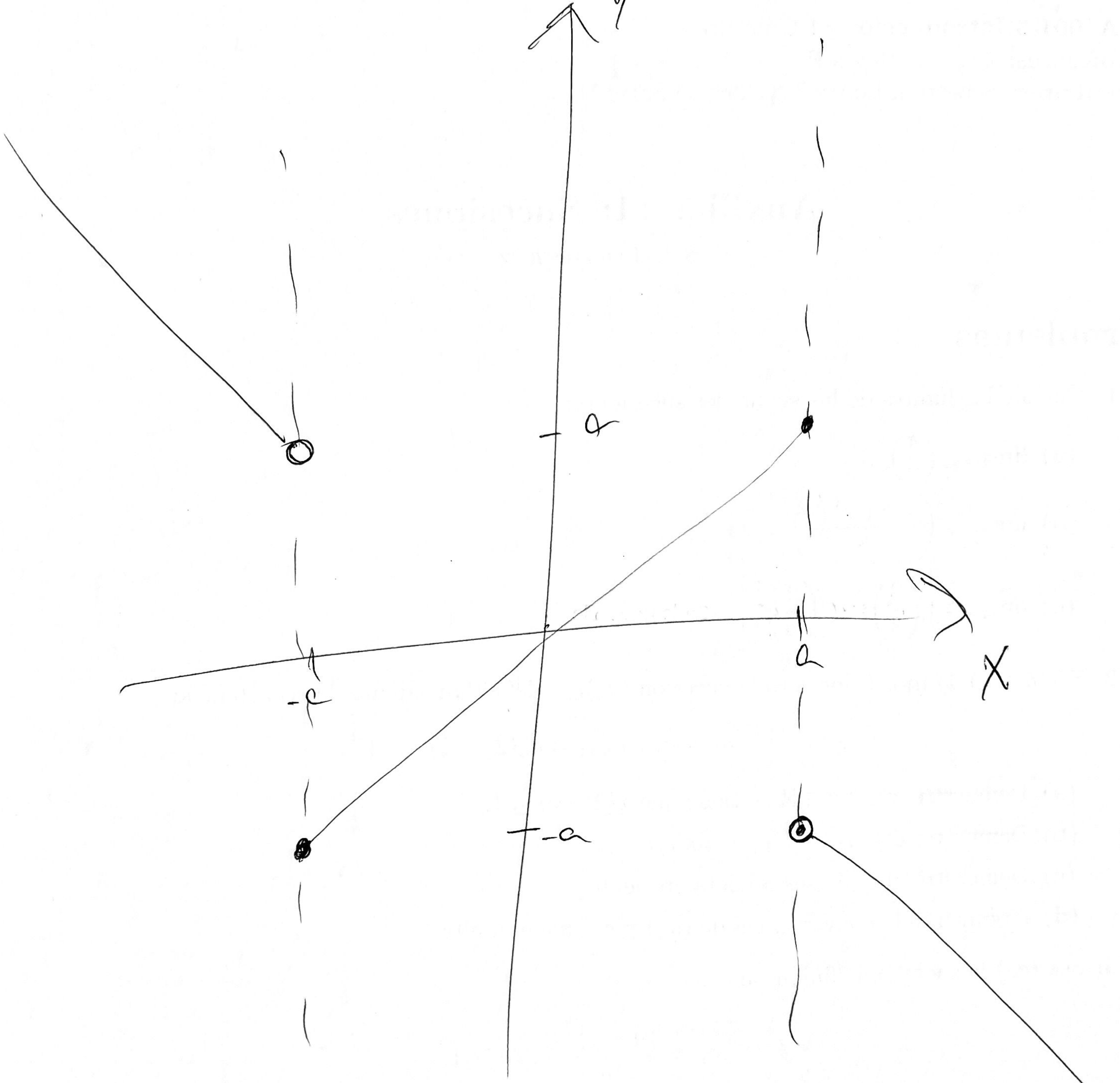
$= 2 \int_{0}^{a} x^2 f_X(x)\,dx - 2 \int_{a}^{+\infty} x^2 f_X(x)\,dx$

P3) b) $= 2\left(\int_0^a x^2 f_X(x)\,dx - \int_a^{+\infty} x^2 f_X(x)\,dx \right)$

$$= 2\left(\int_0^{+\infty} x^2 f_X(x)\,dx - 2\int_a^{+\infty} x^2 f_X(x)\,dx \right)$$

$$= 2\int_0^{+\infty} x^2 f_X(x)\,dx - 4\int_a^{+\infty} x^2 f_X(x)\,dx$$

$$= Var(X) - 1 = 1 - 1 = 0$$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