Pauta Aux Extra (Con Poca explicación)

P1 a)

$$\lim_{N\to\infty} \frac{1}{N}\sum_{K=1}^{N} \ln\left(1+\frac{1}{K}\right)$$

$$= \lim_{N\to\infty} \frac{1}{N}\sum_{K=1}^{N} \ln(K+1) - \ln(K)$$

Prop $\ln(\,)$

Telescopico

$$= \lim_{N\to\infty} \frac{1}{N}\left(\ln(N+1) - \cancel{\ln(1)}^{\,0}\right)$$

$$= \lim_{N\to\infty} \ln\left(\sqrt[N]{N+1}\right) \underset{\ln(a_n)\to\ln(L)}{\overset{☆}{=}} \ln(1)$$

☆ $\sqrt[N]{N+1} \to 1$

$$\underset{\downarrow\, 1}{\sqrt[N]{N}} \le \sqrt[N]{N+1} \le \sqrt[N]{2N} = \sqrt[N]{2}\cdot\sqrt[N]{N} \underset{\searrow\, 1}{}$$

$\Longrightarrow \sqrt[N]{N+1} \to 1$ por Sandwich

b) $\lim_{n\to\infty} \left(\frac{n+5}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{5}{n}\right)^n = e^5$

pues $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$

c) $\lim_{n\to\infty} \sqrt[n]{8^n + n^{2000}\cdot 7^n + e^n}$

$$8 \leq \sqrt[n]{8^n+n^{2000}\cdot 7^n+e^n} \leq \sqrt[n]{3\cdot 8^n}$$

$$\underbrace{\sqrt[n]{3}}_{\to 1}\cdot 8$$

Por Sandwich $\sqrt[n]{8^n+n^{2000}\,7^n+e^n} \longrightarrow 8$

d) Hacer suma desde $\boxed{k=0}$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n} e^{\frac{k}{n}} = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n}\left(e^{\frac{1}{n}}\right)^k$$

$$= \lim_{n\to\infty} \frac{1}{n} \frac{e^{\frac{n+1}{n}}-1}{e^{\frac{1}{n}}-1}$$

$$= \lim_{n\to\infty} \left(e^{\frac{n+1}{n}} - 1\right) \cdot \frac{1}{\left(\dfrac{e^{\frac{1}{n}} - 1}{\frac{1}{n}}\right)}$$

Como $\dfrac{e^{a_n} - 1}{a_n} \to e^0 = 1$ si $a_n \to 0$

$$= (e-1)$$

$\boxed{P_2}$ PDQ $S_n \geq 0$

CB: $S_1 = 1 \geq 0$ ✓

H: Suponiendo que $S_n \geq 0$

PDQ: $S_{n+1} \geq 0$, $S_{n+1} = \sqrt{2+S_n} \geq \sqrt{2} \geq 0$

$\Rightarrow$ Por inducción $S_n \geq 0$

PDQ $S_n \leq 2$

CB: $S_1 = 1 \leq 2$

H: Suponiendo que $S_n \leq 2$

PDQ: $S_{n+1} \leq 2$, $S_{n+1} = \sqrt{2+S_n} \leq \sqrt{2+2} = 2$

$\Rightarrow S_{n+1} \leq 2$

$\Rightarrow$ Por inducción $S_n \leq 2$

<u>Crecimiento</u>

$$S_{n+1} - S_n = \sqrt{2+S_n} - S_n = \frac{2+S_n-S_n^2}{\sqrt{2+S_n}+S_n}$$

$$= \frac{\overbrace{(2-S_n)}^{\text{Positivo}}\overbrace{(S_n+1)}^{\text{Positivo}}}{\underbrace{\sqrt{2+S_n}}_{\text{Positivo}} + \underbrace{S_n}_{\text{Positivo}}} \geq 0$$

$\Rightarrow S_{n+1} \geq S_n$

Como S_n creciente y acotado superior

Teorema de convergencia monotona $\Rightarrow S_n$ converge

Como S_n converge S_{n+1} también y $S_n \to l$
$S_{n+1} \to l$

$\Rightarrow l = \sqrt{2+l} \quad /()^2$

$l^2 = 2+l \Rightarrow \cancel{l=-1}$ o $l=2$ ✓

No soluciona
$l = \sqrt{2+l}$

$\boxed{P3}$ a) $\lim_{x \to 1} (1-x) \tan\left(\frac{x\pi}{2}\right)$

$$= \lim_{x \to 1} (1-x) \frac{\sin\left(\frac{x\pi}{2}\right)}{\cos\left(\frac{x\pi}{2}\right)}$$ Listo

$$= \lim_{x \to 1} \sin\left(\frac{x\pi}{2}\right) \cdot \frac{(1-x)}{\cos\left(\frac{x\pi}{2}\right)}$$

$$= \lim_{u \to 0} \sin\left(\frac{u+1}{2} \cdot \pi\right) \cdot \frac{-u}{\cos\left((u+1)\frac{\pi}{2}\right)}$$

$u = x-1$

Usar $\cos(\alpha+\beta)$

$$= \lim_{u \to 0} \sin\left(\frac{u+1}{2} \cdot \pi\right) \cdot \frac{-u}{\sin(u)}$$

$$= \sin\left(\frac{\pi}{2}\right) \cdot (-1) = \boxed{-1}$$

b) $\lim_{x\to\pi} \dfrac{e^{a(x-\pi)} - e^{b(x-\pi)}}{x^2-\pi^2}$

$= \lim_{x\to\pi} \dfrac{e^{a(x-\pi)}-1 - (e^{b(x-\pi)}-1)}{(x-\pi)} \cdot \dfrac{1}{(x+\pi)}$

$= \lim_{x\to\pi} \left(\dfrac{e^{a(x-\pi)}-1}{a(x-\pi)} \cdot a - \dfrac{e^{b(x-\pi)}-1}{b(x-\pi)} \cdot b \right) \dfrac{1}{x+\pi}$

= Usando $u = a(x-\pi)$, $u \to 0$

y $v = b(x-\pi)$, $v \to 0$, junto con $\lim_{u\to 0} \dfrac{e^u-1}{u} = 1$

$= \boxed{\dfrac{(a-b)}{2\pi}}$

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c) $\lim_{x\to 2} \frac{2^x-4}{x-2} = \lim_{x\to 2} \frac{e^{x\cdot\ln(2)} - e^{2\ln(2)}}{x-2}$

$= \lim_{x\to 2} \frac{e^{x\cdot\ln(2)} - e^{2\cdot\ln(2)}}{x\ln(2) - 2\ln(2)} \cdot \ln(2) \cdot$

$= \lim_{u\to 2\cdot\ln(2)} \frac{e^{u} - e^{2\ln(2)}}{u - 2\ln(2)} \cdot \ln(2)$

$u = x\cdot\ln(2)$

Usando que $\lim_{x\to\bar{x}} \frac{e^x - e^{\bar{x}}}{x-\bar{x}} = e^{\bar{x}}$

$= e^{2\ln(2)} \cdot \ln(2) = 2^2\cdot\ln(2)$

$= \boxed{4\ln(2)}$

d) $\lim_{x\to 0} \left(1+\frac{x}{x^2+1}\right)^{\frac{1}{x}} = L$

$\Rightarrow \lim_{x\to 0} \ln\left(\left(1+\frac{x}{x^2+1}\right)^{\frac{1}{x}}\right) = \ln(L)$

$= \lim_{x\to 0} \frac{\ln\left(1+\frac{x}{x^2+1}\right)}{x}$.

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$$= \lim_{x\to 0} \frac{\ln\left(1+\frac{x}{x^2+1}\right)}{\frac{x}{x^2+1}} \cdot \frac{1}{x^2+1}$$

Usando $\lim_{u\to 0} \frac{\ln(1+u)}{u} = 1$

Usando $u = \frac{x}{x^2+1}$, $u \to 0$

$$= 1 \cdot \frac{1}{0+1} = \boxed{1}$$

$\boxed{P4}$ a) Sea $\varepsilon > 0$, Candidato $L = 0$

$$|x \sin(\tfrac{1}{x}) - 0| = |x| \cdot |\sin(\tfrac{1}{x})| \leq |x| \leq \delta$$

tomando $\delta = \varepsilon \to |x \cdot \sin(\frac{1}{x}) - 0| \leq \varepsilon$

si $|x| < \delta$

b) Sea $\varepsilon > 0$, $L = 2$

$$\left|\frac{\sqrt{3+x}}{x} - 2\right| = \left|\frac{\sqrt{3+x} - 2x}{x}\right| = \left|\frac{3+x-4x^2}{x(\sqrt{3+x}+2x)}\right| = \frac{|-(x-1)(4x+3)|}{|x(\sqrt{3+x}+2x)|}$$

$$= \frac{|x-1||4x+3|}{|x||\sqrt{3+x}+2x|} \leq \delta \frac{|4x+3|}{|x||\sqrt{3+x}+2x|}$$

Suponiendo $\delta < 1/2 \Rightarrow -\frac{1}{2} < x - 1 < 1/2$

$$\Rightarrow \frac{1}{2} < x < \frac{3}{2}$$

$$\Rightarrow 2 > \frac{1}{x} > \frac{2}{3}$$

Como $x > 0 \Rightarrow \sqrt{3+x} + 2x > 1$

$$\Rightarrow \frac{1}{\sqrt{3+x} + 2x} < 1$$

Por otro lado $2 < 4x < 6 \Rightarrow 5 < 4x + 3 < 9$

$$\Rightarrow \delta \frac{|4x+3|}{|x|\,|\sqrt{3+x}+2x|} < \delta \cdot \frac{2 \cdot 1 \cdot 9}{18}$$

Finalmente tomando $\delta = \min\{\frac{\varepsilon}{18}, 1/2\}$

$$\Rightarrow \left| \frac{\sqrt{3+x}}{x} - 2 \right| \leq 18\delta < \varepsilon$$

Se tiene