Tarea 1

P1

a) Para calcular la fuerza, primero necesitamos calcular el campo eléctrico en el eje $\hat{k}$

- $\vec{r} = z\hat{k}$
- $\vec{r}' = \rho'\hat{\rho}'$
- $\vec{r}-\vec{r}' = z\hat{k} - \rho'\hat{\rho}'$
- $|\vec{r}-\vec{r}'| = (z^2+\rho'^2)^{1/2}$
- $\sigma(r') = -qh/2\pi(\rho'^2+h^2)^{3/2}$
- $ds' = \rho' d\phi' d\rho'$

Por lo que el campo sería

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{\infty} \frac{-qh}{2\pi(\rho'^2+h^2)^{3/2}} \frac{z\hat{k}-\rho'\hat{\rho}'}{(z^2+\rho'^2)^{3/2}}\rho' d\rho' d\phi'$$

por integral en ϕ'

$$= \frac{-qh}{8\pi^2\varepsilon_0}\left[z\hat{k}\int_0^{2\pi}\int_0^{\infty} \frac{\rho' d\rho' d\phi'}{(\rho'^2+h^2)^{3/2}(z^2+\rho'^2)^{3/2}} - \cancel{\int_0^{2\pi}\int_0^{\infty} \frac{-\rho'^2(\cos\phi'\hat{\imath}+\sin\phi\hat{\jmath})}{(\rho'^2+h^2)^{3/2}(z^2+\rho'^2)^{3/2}} d\rho' d\phi'}\right]$$

$$= -\frac{qhz\hat{k}}{8\pi^2\varepsilon_0} 2\pi \int_0^{\infty} \frac{\rho' d\rho'}{(\rho'^2+h^2)^{3/2}(z^2+\rho'^2)^{3/2}}$$

$$= -\frac{q\cancel{hz}\hat{k}}{4\pi\varepsilon_0} \cdot \frac{1}{\cancel{hz}(z+h)^2} = -\frac{1}{4\pi\varepsilon_0}\frac{q}{(z+h)^2}\hat{k}$$

Como la partícula está en $z=h$, la fuerza sería

$$\vec{F} = q\vec{E}(h\hat{k}) = -\frac{1}{16\pi\varepsilon_0}\frac{q^2}{h^2}\hat{k}$$

b) La fuerza sobre la carga $\forall z$ es $\vec{F}(z\hat{k}) = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(z+h)^2}\hat{k}$, por lo que la ec. de mov. es

$$m\ddot{z}\hat{k} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(z+h)^2}\hat{k}$$

truco de mecánica

$$\Rightarrow m\int_{\dot{z}_0}^{\dot{z}} \dot{z}\,d\dot{z} = -\frac{q^2}{4\pi\varepsilon_0}\int_{z_0}^{z}\frac{dz}{(z+h)^2}$$

donde las condiciones iniciales son $\dot{z}(t=0)=0$ y $z(t=0)=h$.

$$\Leftrightarrow m\frac{\dot{z}^2}{2} = -\frac{q^2}{4\pi\varepsilon_0}\cdot\left(-\frac{1}{z+h}\right)\Big|_h^z$$

$$\Leftrightarrow m\frac{\dot{z}^2}{2} = \frac{q^2}{4\pi\varepsilon_0}\left(\frac{1}{z+h}-\frac{1}{2h}\right)$$

y nos interesa la velocidad en $z=0$ $\Rightarrow \dot{z}(z=0) = \sqrt{\frac{q^2}{2m\pi\varepsilon_0}\frac{1}{2h}}$

P2

a) La Ley de Gauss en su forma diferencial es $\vec{\nabla}\vec{E} = \rho/\varepsilon_0$, por lo que debemos calcular la divergencia en coordenadas cartesianas

$$\vec{\nabla}\cdot\vec{E}(\bar{r}) = \frac{\partial E(z)}{\partial z} = -\frac{\partial}{\partial z}\left(ae^{-\alpha z} + be^{-\beta z}\right) = a\alpha e^{-\alpha z} + b\beta e^{-\beta z}$$

$$\Rightarrow \rho(z) = a\alpha\varepsilon_0 e^{-\alpha z} + b\beta\varepsilon_0 e^{-\beta z}$$

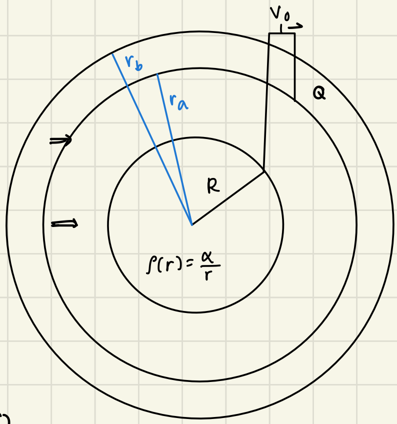
b) Para la carga total ocupamos $Q = \int \rho(r')d\tau'$. Consideremos un paralelepípedo donde $A = \int dxdy$ y $z \in [0,\infty)$

$$\Rightarrow Q = \underbrace{\iint}_{A}\int_0^\infty \left(a\alpha\varepsilon_0 e^{-\alpha z'} + b\beta\varepsilon_0 e^{-\beta z'}\right)dx'dy'dz' = A\left[a\alpha\varepsilon_0\int_0^\infty e^{-\alpha z'}dz' + b\beta\varepsilon_0\int_0^\infty e^{-\beta z'}dz'\right]$$

$$= A\left[-a\varepsilon_0 e^{-\alpha z'}\Big|_0^\infty - b\varepsilon_0 e^{-\beta z'}\Big|_0^\infty\right]$$

$$= A\left(a\varepsilon_0 + b\varepsilon_0\right)$$

Pauta P3 tarea 1



→ Dentro del conductor, $\vec{E}=0 \Rightarrow V = cte$

· En esféricas $dV = r^2 \sin\theta \, dr \, d\phi \, d\theta$

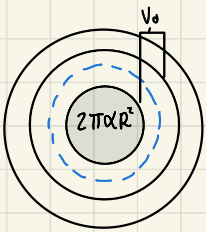
ⓐ

→ $q_{esf.} = \int dq = \int \rho(r)\, dV = \int_0^{\pi}\int_0^{2\pi}\int_0^{R} \frac{\alpha}{r} \cdot r^2 \sin\theta \, dr \, d\phi \, d\theta$

$$= 2\pi\alpha \int_0^{\pi}\int_0^{R} r \sin\theta \, dr \, d\theta = 2\pi\alpha \left[\frac{r^2}{2}\right]_0^R \left[-\cos\theta\right]_0^{\pi} = \pi\alpha R^2[1+1]$$

$$= \boxed{2\pi\alpha R^2}$$

Aplicando ley de Gauss.



$$\int \vec{E}(R<r<r_a)\cdot d\vec{s} = \frac{2\pi\alpha R^2}{\varepsilon_0}$$

$$\Rightarrow \vec{E}(R<r<r_a)\cdot \underbrace{\int_0^{2\pi}\int_0^{\pi} r^2 \sin\theta \, d\theta \, d\phi}_{} = \frac{2\pi\alpha R^2}{\varepsilon_0}$$

$$\Rightarrow \vec{E}(R<r<r_a)\cdot \overset{2}{\cancel{4\pi}} r^2 = \frac{\cancel{2\pi}\alpha R^2}{\varepsilon_0} \Rightarrow \boxed{\vec{E}(R<r<r_a) = \frac{\alpha R^2}{2\varepsilon_0 r^2}}$$

como $\vec V(r) = -\int_{r_0}^{r} \vec E(r)\cdot d\vec r$

$$\underbrace{\vec V(r_a) - \vec V(R)}_{V_0} = -\int_{r_0}^{r_a} \vec E(r)\cdot d\vec r + \int_{r_0}^{R} \vec E(r)\, d\vec r.$$

$$V_0 = \int_{r_a}^{R} \vec E(r)\cdot d\vec r = \int_{r_a}^{R} \frac{\alpha R^2}{2\varepsilon_0 r^2}\, dr = \frac{\alpha R^2}{2\varepsilon_0}\int_{r_a}^{R} \frac{1}{r^2}\, dr$$

$$\Rightarrow V_0 = \frac{\alpha R^2}{2\varepsilon_0}\left(\frac{-1}{r}\right)\Big|_{r_a}^{R} = -\frac{\alpha R^{\cancel{2}}}{2\varepsilon_0 \cancel{R}} + \frac{\alpha R^2}{2\varepsilon_0 r_a}$$

$$\Rightarrow V_0 = \alpha\left(\frac{R^2}{2\varepsilon_0 r_a} - \frac{R}{2\varepsilon_0}\right) = \alpha\left(\frac{R^2 - R r_a}{2\varepsilon_0 r_a}\right)$$

$$\Rightarrow \boxed{\frac{V_0\cdot 2\varepsilon_0 r_a}{R^2 - R\cdot r_a} = \alpha}$$

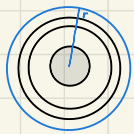
(b)

Aplicando ley de Gauss:

· Encerrando por fuera del cilindro:

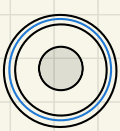
$$\int \vec E(r>r_b)\, d\vec s = \frac{Q + 2\pi\alpha R^2}{\varepsilon_0} \Rightarrow E(r>r_b)\cdot \int_0^{2\pi}\int_0^{\pi} r^2 \operatorname{sen}\theta\, d\theta\, d\phi = \frac{Q + 2\pi\alpha R^2}{\varepsilon_0}$$

$$\Rightarrow \boxed{\vec E(r>r_b) = \frac{Q + 2\pi\alpha R^2}{4\pi r^2 \varepsilon_0}\hat r} \Rightarrow \vec E(r>r_0) = \frac{Q + 2\pi\left(\frac{V_0\cdot 2\varepsilon_0 r_a}{R^2 - R r_a}\right) R^2}{4\pi r^2 \varepsilon_0}\hat r$$



· Dentro del cascarón:

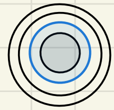
$\boxed{\vec E(r_a<r<r_b) = 0}$ → Debido a que el cascarón esférico es un conductor.



· Entre la esfera y el cascarón:

$$\boxed{\vec{E}(R<r<r_a) = \frac{\alpha R^2}{2\varepsilon_0 r^2}\hat{r}}$$

→ Calculado anteriormente.

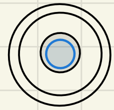


$$\Rightarrow \quad \vec{E}(R<r<r_a) = \frac{\left(\frac{V_0 \cdot 2\varepsilon_0 r_a}{R^2 - R\cdot r_a}\right) R^2}{2\varepsilon_0 r^2}\hat{r}$$

Dentro de la esfera:

$$\cdot \int \vec{E}(r<R)\cdot d\vec{s} = \pi \alpha r^2 2 \quad \Rightarrow \quad E(r<R)\cdot \cancel{4}^{2}\cancel{\pi}\cancel{r^2} = \frac{\cancel{\pi}\alpha \cancel{r^2}\cdot \cancel{2}}{\varepsilon_0}$$

$$\Rightarrow \quad \boxed{\vec{E}(r<R) = \frac{\alpha}{2\varepsilon_0}\hat{r}}$$



$$\Rightarrow \quad \vec{E}(r<R) = \frac{V_0 \cdot \cancel{2\varepsilon_0} r_a}{(R^2 - R\cdot r_a)\cancel{2\varepsilon_0}}\hat{r} = \frac{V_0 r_a}{(R^2 - R r_a)}\hat{r}$$

ⓒ Fuera del cilindro:

$$\cdot \quad V(r>r_b) = -\int_{r_0}^{r} \vec{E}(r>r_0)\cdot d\vec{r} = -\int_{r_0}^{r} \frac{Q + 2\pi\alpha R^2}{4\pi r^2 \varepsilon_0}\, dr$$

$V(r_0) = 0$

$$V(r>r_b) = -\frac{Q + 2\pi\alpha R^2}{4\pi\varepsilon_0}\cdot\left[-\frac{1}{r}\right]_{r_0}^{r} = \frac{Q + 2\pi\alpha R^2}{4\pi\varepsilon_0 r} - \cancel{\frac{Q + 2\pi\alpha R^2}{4\pi\varepsilon_0 r_0}}$$

$$\Rightarrow \quad \boxed{V(r>r_0) = \frac{Q + 2\pi\alpha R^2}{4\pi\varepsilon_0 r}}$$

En el conductor

$$\cdot \quad V(r_a<r<r_b) = \frac{Q + 2\pi\alpha R^2}{4\pi\varepsilon_0 r_b}$$

$$\Rightarrow \quad \boxed{V(r_a<r<r_b) = \frac{Q + 2\pi\alpha R^2}{4\pi\varepsilon_0 r_b}}$$

→ Es cte ya que es un conductor. V se mantiene desde $r = r_b$.

Entre la esfera y el conductor

$$\cdot \quad V(R<r<r_a) = -\int_{r_a}^{r} \frac{\alpha R^2}{2\varepsilon_0 r^2}\, dr + \frac{2\pi\alpha R^2 + Q}{4\pi\varepsilon_0 r_b} = -\frac{\alpha R^2}{2\varepsilon_0}\left[-\frac{1}{r}\right]_{r_a}^{r} + \frac{2\pi\alpha R^2 + Q}{4\pi\varepsilon_0 r_b}$$

$$\Rightarrow \quad \boxed{V(R<r<r_a) = \frac{\alpha R^2}{2\varepsilon_0}\left(\frac{1}{r} - \frac{1}{r_a}\right) + \frac{2\pi\alpha R^2 + Q}{4\pi\varepsilon_0 r_b}}$$

Dentro de la esfera

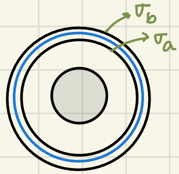
$$V(r<R) = -\int_R^r \frac{\alpha}{2\varepsilon_0}\,dr + \frac{\alpha R^2}{2\varepsilon_0}\left(\frac{1}{R}-\frac{1}{r_a}\right) + \frac{2\pi\alpha R^2 + Q}{4\pi\varepsilon_0 r_b}$$

$$= -\frac{\alpha}{2\varepsilon_0}\left[r\right]_R^r + \frac{\alpha R^2}{2\varepsilon_0}\left(\frac{1}{R}-\frac{1}{r_a}\right) + \frac{2\pi\alpha R^2 + Q}{4\pi\varepsilon_0 r_b}$$

$$= \frac{\alpha}{2\varepsilon_0}(R-r) + \frac{\alpha R^2}{2\varepsilon_0}\left(\frac{1}{R}-\frac{1}{r_a}\right) + \frac{2\pi\alpha R^2 + Q}{4\pi\varepsilon_0 r_b}$$

$$\Rightarrow \boxed{V(r<R) = \frac{\alpha R}{\varepsilon_0} - \frac{\alpha r}{2\varepsilon_0} - \frac{\alpha R^2}{2\varepsilon_0 r_a} + \frac{2\pi\alpha R^2 + Q}{4\pi\varepsilon_0 . r_b}}$$

(d) para $r = r_a$:



→ En el conductor, la carga se distribuye en la superficie.

$$\int \cancel{\vec{E}}^{\,0}\, d\vec{s} = \frac{\int \sigma_a\, ds + q_{esf}}{\varepsilon_0}$$

$$0 = \frac{\sigma_a \cdot \cancel{4}^{\,2}\cancel{\pi}\, r_a^2 + \cancel{2}\cancel{\pi}\alpha R^2}{\cancel{\varepsilon_0}}$$

$$\Rightarrow \boxed{-\frac{\alpha R^2}{2 r_a^2} = \sigma_a} \Rightarrow \sigma_a = -\frac{V_0\cdot\cancel{2}\varepsilon_0\cancel{r_a}}{R^{\cancel{2}} - \cancel{R}\cdot r_a}\cdot\frac{R^{\cancel{2}}}{\cancel{2}\, r_a^{\cancel{2}}} \Rightarrow \boxed{\sigma_a = \frac{V_0\cdot\varepsilon_0}{(r_a-R)}\cdot\frac{R}{r_a}}$$

$$\Rightarrow \int \sigma_b\, ds + \int \sigma_a\, ds = Q$$

$$\Rightarrow \sigma_b\, 4\pi r_b^2 + \frac{V_0\varepsilon_0}{(r_a-R)}\cdot\frac{R}{\cancel{r_a}}\cdot 4\pi r_a^{\cancel{2}} = Q$$

$$\Rightarrow \sigma_b\, \cancel{4}\cancel{\pi}\, \cancel{r_b^2} = \frac{Q}{4\pi r_b^2} - \frac{V_0\varepsilon_0 R r_a\cdot\cancel{4\pi}}{(r_a-R)\, r_b^2}$$

$$\Rightarrow \boxed{\sigma_b = \frac{Q}{4\pi r_b^2} - \frac{V_0\varepsilon_0 R\cdot r_a}{(r_a-R)\, r_b^2}}$$

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