# Auxiliar #13

p1. $\left\langle \underset{u_1}{\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}}, \underset{u_2}{\begin{pmatrix}0\\1\\0\\1\end{pmatrix}}, \underset{u_3}{\begin{pmatrix}0\\0\\1\\1\end{pmatrix}} \right\rangle$ base de $U$

a) base ortonormal de $U$. El algoritmo es

$$\tilde{u}_1 = \frac{u_1}{\|u_1\|} \quad \text{donde} \quad \|u_1\| = \sqrt{\langle u_1, u_1 \rangle}$$

Calculemos $\|u_1\| = \sqrt{1\cdot 1 + 0\cdot 0 + 0\cdot 0 + -1\cdot -1}$

$$= \sqrt{2}$$

$$\Rightarrow \tilde{u}_1 = \begin{pmatrix}1/\sqrt{2}\\0\\0\\-1/\sqrt{2}\end{pmatrix} \qquad U_1 = \langle \tilde{u}_1 \rangle$$

$$\overline{u}_2 = u_2 - P_{U_1}(u_2) \quad y \quad P_{U_1}(u_2) = \langle u_2, \tilde{u}_1 \rangle \cdot \tilde{u}_1$$

$$= \begin{pmatrix}0\\1\\0\\1\end{pmatrix} - \left(1/\sqrt{2}\cdot 0 + 1\cdot 0 + 0\cdot 0 + 1\cdot -1/\sqrt{2}\right)\cdot \begin{pmatrix}1/\sqrt{2}\\0\\0\\-1/\sqrt{2}\end{pmatrix}$$

$$= \begin{pmatrix}0\\1\\0\\1\end{pmatrix} - \left(-1/\sqrt{2}\right)\begin{pmatrix}1/\sqrt{2}\\0\\0\\-1/\sqrt{2}\end{pmatrix} = \begin{pmatrix}0\\1\\0\\1\end{pmatrix} - \begin{pmatrix}-1/2\\0\\0\\1/2\end{pmatrix} = \begin{pmatrix}1/2\\1\\0\\1/2\end{pmatrix}$$

falta normalizar $\tilde{u}_2 = \dfrac{\overline{u}_2}{\|\overline{u}_2\|} = \begin{pmatrix}1/2\\1\\0\\1/2\end{pmatrix} 1/\sqrt{1/4+1+0+1/4}$

$$= \sqrt{2/3}\cdot 1/2 \begin{pmatrix}1\\2\\0\\1\end{pmatrix}$$

$$= 1/\sqrt{6}\begin{pmatrix}1\\2\\0\\1\end{pmatrix}$$

luego $U_1 = \langle \tilde{u}_1, \tilde{u}_2 \rangle$

luego;

$$\overline{u}_3 = u_3 - P_{U_2}(u_3)$$

$$= u_3 - \left( \langle u_3, \tilde{u}_2 \rangle \tilde{u}_2 + \langle u_3, \tilde{u}_1 \rangle \tilde{u}_1 \right)$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} -$$

i) $\langle u_3, \tilde{u}_2 \rangle = 0 \cdot 1/\sqrt{6} \cdot 1 + 0 \cdot 2/\sqrt{6} + 1 \cdot 0 + 1 \cdot 1/\sqrt{6}$

$= 1/\sqrt{6}.$

$\left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, 1/\sqrt{6} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\rangle$

$= 1/\sqrt{6}\,(0 \cdot 1 + 0 \cdot 2 + 1 \cdot 0 + 1 \cdot 1)$

ii) $\langle u_3, \tilde{u}_1 \rangle = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, 1/\sqrt{2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right\rangle = 1/\sqrt{2}\,(0 \cdot 1 + 0 \cdot 0 + 1 \cdot 0 + 1 \cdot -1)$

$= -1/\sqrt{2}$

$\therefore$

$$\overline{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \left[ 1/\sqrt{6} \cdot 1/\sqrt{6} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} + (-1/\sqrt{2}) \cdot 1/\sqrt{2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \left[ \begin{pmatrix} 1/6 \\ 2/6 \\ 0 \\ 1/6 \end{pmatrix} + \begin{pmatrix} -1/2 \\ 0 \\ 0 \\ 1/2 \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} -2/6 \\ 2/6 \\ 0 \\ 4/6 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ -1/3 \\ 1 \\ 1/3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ -1 \\ 3 \\ 1 \end{pmatrix}$$

Luego normalizamos

$$\tilde{u}_3 = \frac{\bar{u}_3}{\|\bar{u}_3\|}$$

donde

$$\|\bar{u}_3\| = \sqrt{\left\langle \frac{1}{3}\begin{pmatrix} 1 \\ -1 \\ 3 \\ 1 \end{pmatrix}, \frac{1}{3}\begin{pmatrix} 1 \\ -1 \\ 3 \\ 1 \end{pmatrix} \right\rangle}$$

$$= \frac{1}{3}\sqrt{1\cdot 1 + -1\cdot -1 + 3\cdot 3 + 1\cdot 1}$$

$$= \frac{1}{3}\sqrt{1+1+9+1}$$

$$= \frac{1}{3}\sqrt{12}$$

$$= \frac{2}{3}\sqrt{3}$$

$$= \frac{2}{\sqrt{3}}$$

$$\therefore\ \tilde{u}_3 = \frac{1/3}{2/\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 3 \\ 1 \end{pmatrix}$$

$$= \frac{\sqrt{3}}{2\cdot 3}\begin{pmatrix} 1 \\ -1 \\ 3 \\ 1 \end{pmatrix} = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 3 \\ 1 \end{pmatrix}$$

ortogonal

$$\therefore\ U = \left\langle \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 3 \\ 1 \end{pmatrix} \right\rangle$$

b) Ahora hay que encontrar $U^{\perp}$ su ortonormal.

usamos que; $\mathbb{R}^n = W \oplus W^{\perp}$

como están en $\oplus$; $\dim(\mathbb{R}^4) = \dim(U) + \dim(U^{\perp})$

$$4 = 3 + 1$$

es solo 1 vectorcito!

$$U^{\perp} = \{ v \in \mathbb{R}^n / \langle u, v \rangle = 0 \ \forall u \in U \}$$

Como escribo un $u \in U$ cualquiera?

$$u = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$\therefore \quad v \in U^{\perp} \quad ; \quad v = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$

$$\left\langle \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}, \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle = \emptyset$$

$$= \alpha \underbrace{\left\langle \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right\rangle} + \beta \underbrace{\left\langle \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle} + \gamma \underbrace{\left\langle \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle} = 0$$

$\therefore \quad (1) = 0 \qquad (2) = 0 \qquad (3) = 0$

(1) $x - w = 0$ (2) $y + w = 0$ (3) $z + w = 0$

$\boxed{x = w} \qquad \boxed{y = -w} \quad y \quad \boxed{z = -w}$

$$\therefore v = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} w \\ -w \\ -w \\ w \end{pmatrix} = w \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

∴ base de $U^{\perp}$ es $\left\langle \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \right\rangle$

al ser solo 1 elemento la base ortonormal de $U^{\perp}$ sería

$$\tilde{V}_1 = \frac{\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}}{\|V_1\|} \quad \text{donde } \|V_1\| = \sqrt{1\cdot 1 + -1\cdot -1 + -1\cdot -1 + 1\cdot 1}$$
$$= \sqrt{4} = 2$$

$$= 1/2 \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \quad \therefore \quad \tilde{U}^{\perp} = \left\langle 1/2 \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \right\rangle$$

p2)

a) pol caract y vp.

$$p(\lambda) = |A - \lambda I| = \left| \begin{pmatrix} 5-\lambda & 0 & -2 \\ 0 & 7-\lambda & -2 \\ -2 & -2 & 6-\lambda \end{pmatrix} \right|$$

$$= (5-\lambda) \begin{vmatrix} (7-\lambda) & -2 \\ -2 & (6-\lambda) \end{vmatrix} + -2 \begin{vmatrix} 0 & 7-\lambda \\ -2 & -2 \end{vmatrix}$$

$$= (5-\lambda)\left[(7-\lambda)(6-\lambda) - 4\right] + -2\left(-(7-\lambda)(-2)\right)$$

$$= (5-\lambda)(7-\lambda)(6-\lambda) - 4(5-\lambda) - 2(14 - 2\lambda)$$

$$= (5-\lambda)(7-\lambda)(6-\lambda) - 20 + 4\lambda - 28 + 4\lambda$$
$$= (5-\lambda)(7-\lambda)(6-\lambda) - 48 + 8\lambda$$

$(x+a)(x+b) = x^2 + (a+b)x + ab$

$= (5-\lambda)(7-\lambda)(6-\lambda) - 8(6-\lambda)$

$= (6-\lambda)\left[(5-\lambda)(7-\lambda) - 8\right]$

$= (6-\lambda)\left[35 - 5\lambda - 7\lambda + \lambda^2 - 8\right]$

$= (6-\lambda)\left[27 - 12\lambda + \lambda^2\right]$

$= (6-\lambda)(\lambda-3)(\lambda-9)$

$\therefore\ p(\lambda) = (6-\lambda)(\lambda-3)(\lambda-9)$

Luego los vp $\Rightarrow$ $\lambda_1 = 6$
$\lambda_2 = 3$
$\lambda_3 = 9$

b) $\vec{vep}_i$

$\lambda_1 = 6$ $\quad (A - \lambda_1 I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & -2 \\ -2 & -2 & 0 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \begin{aligned} &1)\ -x - 2z = 0 \\ &2)\ y - 2z = 0 \\ &3)\ -2x - 2y = 0 \end{aligned}$$

de 1) $x = -2z$ luego en 2) $y = 2z$

$\therefore$ en 3) $-2(-2z) - 2(2z) = 0$

$\boxed{4z - 4z = 0}$ no hay más info

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2z \\ 2z \\ z \end{pmatrix} = z \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \quad \therefore W_{\lambda_1=6} = \left\langle \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \right\rangle$$

$\boxed{\lambda_2 = 3}$

$$\begin{pmatrix} 2 & 0 & -2 \\ 0 & 4 & -2 \\ -2 & -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

1) $2x - 2z = 0 \longrightarrow \boxed{x = z}$

2) $4y - 2z = 0 \longrightarrow \boxed{y = 1/2 z}$

3) $-2x - 2y + 3z = 0$

en 3); $-2z - z + 3z = 0$ no más info

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ 1/2 z \\ z \end{pmatrix} = z \begin{pmatrix} 1 \\ 1/2 \\ 1 \end{pmatrix}$$

$$\therefore W_{\lambda_2=3} \left\langle \begin{pmatrix} 1 \\ 1/2 \\ 1 \end{pmatrix} \right\rangle$$

$\boxed{\lambda_3 = 9}$

$$\begin{pmatrix} -4 & 0 & -2 \\ 0 & -2 & -2 \\ -2 & -2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

1) $-4x - 2z = 0$

2) $-2y - 2z = 0$

3) $-2x - 2y - 3z = 0$

de 2) $\boxed{y = -z}$ en 1) $\boxed{x = -1/2 z}$

luego en 3) $z + 2z - 3z = 0$ no hay más info

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1/2 z \\ -z \\ z \end{pmatrix} = z \begin{pmatrix} -1/2 \\ -1 \\ 1 \end{pmatrix} \quad \therefore W_{\lambda_3=9} = \left\langle \begin{pmatrix} -1/2 \\ -1 \\ 1 \end{pmatrix} \right\rangle$$

¿Es posible diagonalizar? Si ya que

$$\underbrace{\alpha(\lambda_1)}_{1} = \underbrace{\gamma(\lambda_1)}_{1} \quad \underbrace{\alpha(\lambda_2)}_{1} = \underbrace{\gamma(\lambda_2)}_{1}$$

$$\underbrace{\alpha(\lambda_3)}_{1} = \underbrace{\gamma(\lambda_3)}_{1}$$

$$A = \begin{bmatrix} -2 & 1 & -1/2 \\ 2 & 1/2 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} -2 & 1 & -1/2 \\ 2 & 1/2 & -1 \\ 1 & 1 & 1 \end{bmatrix}^{-1}$$

c) Es posible ya que la matriz es simetrica

Luego por propiedad que nos dice que vectores propios de valores prop. son distintos son ortogonales ∴ entre ellos estamos bien,

luego solo falta normalizar, es decir,

$$\frac{v_1}{\|v_1\|} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \frac{1}{\sqrt{4+4+1}} = 1/3 \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$$

$$\frac{V_2}{\|V_2\|} = \frac{\begin{pmatrix} 1 \\ 1/2 \\ 1 \end{pmatrix}}{\|V_2\|} \quad \text{donde } \|V_2\| = \sqrt{\left\langle 1/2\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, 1/2\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}\right\rangle}$$

$$= 1/2\sqrt{4+1+4}$$

$$= 3/2$$

$$= \frac{2}{3}\begin{pmatrix} 1 \\ 1/2 \\ 1 \end{pmatrix} = 1/3\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$\frac{V_3}{\|V_3\|} = \frac{\begin{pmatrix} -1/2 \\ -1 \\ 1 \end{pmatrix}}{\|V_3\|} \quad \text{donde } \|V_3\| = \sqrt{\left\langle 1/2\begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}, 1/2\begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}\right\rangle}$$

$$= 1/2\sqrt{1+4+4}$$

$$= \frac{3}{2}$$

$$= \frac{2}{3}\begin{pmatrix} -1/2 \\ -1 \\ 1 \end{pmatrix} = 1/3\begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}$$

$$\therefore \quad A = \underbrace{\begin{bmatrix} -2/3 & 2/3 & -1/3 \\ 2/3 & 1/3 & -2/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 9 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} -2/3 & 2/3 & 1/3 \\ 2/3 & 1/3 & 2/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}}_{P^t} ;$$

d) ¿Es invertible A?

si los vp son no nulos $\Rightarrow$ A es invertible

como $\lambda_1 \neq 0 \neq \lambda_2$
$0 \neq \lambda_3$ $\Rightarrow$ A es invertible