P_1

a) Exprese en forma cartesiana:

1) $(1-i)^4(1+i)^4 = ((1-i)(1+i))^4$

Si tomamos $z = 1+i \Rightarrow \bar{z} = 1-i$

$\Rightarrow z \cdot \bar{z} = |z|^2$

Con esto: $((1-i)(1+i))^4 = (|1+i|^2)^4$

$= (1^2+1^2)^4$

$= 2^4$

b) $1+i+\dfrac{i-1}{|1-i|^2+i}$

Tenemos que $|1-i|^2 = 1^2+1^2 = 2$, con esto:

$$1+i+\frac{i-1}{|1-i|^2+i} = 1+i+\frac{i-2}{2+i}\cdot\frac{2-i}{2-i}$$

$$= 1+i-\frac{(2-i)^2}{4-i^2}$$

$$= 1+i-\frac{4-4i+i^2}{4-1}$$

$$= 1+i-\frac{4-4i-1}{5}$$

$$= 1+i-\frac{3-4i}{5} = 1-\frac{3}{5}+i\left(1+\frac{4}{5}\right)$$

b] Exprese $\frac{1+i\sqrt{3}}{2}$ en su forma polar y pruebe que

$$\left(\frac{1+i\sqrt{3}}{2}\right)^m + \left(-\frac{1+i\sqrt{3}}{2}\right)^m = 2 \iff 6 \mid m$$

Dem]

Primero escribamos en polares:

$$\frac{1+i\sqrt{3}}{2} = \underset{\underset{\cos\alpha}{\uparrow}}{\frac{1}{2}} + i\underset{\underset{\operatorname{sen}\alpha}{\uparrow}}{\frac{\sqrt{3}}{2}} =$$

$\Rightarrow \alpha = \frac{\pi}{3}$ cumple $\operatorname{sen}\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos\frac{\pi}{3} = \frac{1}{2}$

así $e^{i\frac{\pi}{3}} = \cos\frac{\pi}{3} + i\operatorname{sen}\frac{\pi}{3} = \frac{1}{2} + \frac{i\sqrt{3}}{2}$

Veamos la doble implicancia:

$\Leftarrow$] Tenemos que m es divisible por 6

$\Rightarrow \exists k \in \mathbb{N}$, $m = 6k$, reemplazando tenemos

$$\left(e^{i\frac{\pi}{3}}\right)^m + \left(-e^{i\frac{\pi}{3}}\right)^m =$$

$$= e^{i\frac{\pi}{3}m} + (-1)^m \cdot e^{i\frac{\pi}{3}m}$$

$$= e^{i\frac{\pi}{3}\cdot 6k} + (-1)^{6k}\, e^{i\frac{\pi}{3}6k}$$

$$= e^{i\pi\cdot 2k} + e^{i\pi\cdot 2k} = 2\cdot e^{i\pi\cdot 2k}$$

Tenemos que $e^{i2\pi k} = \cos 2\pi k + i\operatorname{sen} 2\pi k$
$= 1 + i\cdot 0 = 1$

$$\Rightarrow \left(e^{i\frac{\pi}{3}}\right)^m + (-1)^m e^{i\frac{\pi}{3}m} = 2\cdot e^{i\pi\cdot 2k} = 2$$

$\Rightarrow$] Veamos que m no puede ser impar. Si $m = 2k+1$

$$\Rightarrow e^{i\frac{\pi}{3}m} + (-1)^m \cdot e^{i\frac{\pi}{3}m}$$

$$= e^{i\frac{\pi}{3}m} - e^{i\frac{\pi}{3}m} = 0 \neq 2$$ $\rightarrow\leftarrow$

$\Rightarrow m$ es par, entonces:

$$\Rightarrow e^{i\frac{\pi}{3}\cdot m} + (-1)^m \cdot e^{i\frac{\pi}{3}m} = 2$$

$$\Rightarrow e^{i\frac{\pi}{3}\cdot m} + e^{i\frac{\pi}{3}\cdot m} = 2$$

$$\Rightarrow 2 \cdot e^{i\frac{\pi}{3}\cdot m} = 2$$

$$\Rightarrow e^{i m\frac{\pi}{3}} = 1 \Rightarrow \exists k \in \mathbb{N} \text{ tq } m\frac{\pi}{3} = 2k\cdot\pi$$

$$\Rightarrow m = 6k$$

c) $\forall n \in \mathbb{N} \quad (1-pe^{i\frac{\pi}{2}})^n + (1+pe^{i\frac{\pi}{2}})^n \in \mathbb{R}$

Dem) Sea $z = (1-pe^{i\frac{\pi}{2}})^n$

Tenemos que:

$$\overline{z} = \overline{(1-pe^{i\frac{\pi}{2}})^n} = \overline{(1-pe^{i\frac{\pi}{2}})}\,\overline{(1-pe^{i\frac{\pi}{2}})} \cdots \overline{(1-pe^{i\frac{\pi}{2}})}$$

$$= (1+pe^{i\frac{\pi}{2}})(1+pe^{i\frac{\pi}{2}}) \cdots (1+pe^{i\frac{\pi}{2}})$$

$$= (1+pe^{i\frac{\pi}{2}})^n$$

Con esto:

$$\Rightarrow (1-pe^{i\frac{\pi}{2}})^n + (1+pe^{i\frac{\pi}{2}})^n = z + \overline{z} \underset{\text{Propiedad}}{=} 2\operatorname{Re}(z) \in \mathbb{R}$$

P2) Considere $A = \sum_{k=0}^{n} \binom{n}{k} \cos(k\alpha)$, $B = \sum_{k=0}^{n} \binom{n}{k} \sin(k\alpha)$

a) Pruebe que $A + iB = (1 + \cos\alpha + i\sin\alpha)^n$

Dem) $$A + iB = \sum_{k=0}^{n} \binom{n}{k} \cos(k\alpha) + i\sum_{k=0}^{n} \binom{n}{k} \sin(k\alpha)$$

$$= \sum_{k=0}^{n} \binom{n}{k} \{\cos(k\alpha) + i\,\mathrm{sen}(k\alpha)\}$$

$$= \sum_{k=0}^{n} \binom{n}{k} e^{ik\alpha} = (1 + e^{i\alpha})^n = (1 + \cos\alpha + i\,\mathrm{sen}\,\alpha)^n$$

b] escriba $1+\cos(\alpha)+i\,\text{sen}(\alpha)$ en su formula y deduzca que

$$A = 2^n \cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{n\alpha}{2}\right), \quad B = 2^n \cos\left(\frac{\alpha}{2}\right)\sin\left(\frac{n\alpha}{2}\right)$$

Dem]

Queremos escribir: $re^{i\beta} = 1+\cos\alpha + i\,\text{sen}\,\alpha$

$\hookrightarrow$ hay que buscar el r y el β.

. Usando el hint tenemos:

$$\cdot \sin(\alpha) = \sin\left(2\cdot\frac{\alpha}{2}\right) = 2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)$$

$$\cdot \cos(\alpha) = \cos\left(2\cdot\frac{\alpha}{2}\right) = \cos^2(\alpha/2) - \sin^2(\alpha/2)$$

$$\Rightarrow 1+\cos\alpha + i\,\text{sen}\,\alpha = 1 + \cos^2\frac{\alpha}{2} - \text{sen}^2\frac{\alpha}{2} + i\cdot 2\sin\alpha/2\,\cos\alpha/2$$

$$= \cos^2\alpha/2 + \cancel{\text{sen}^2\alpha/2} + \cos^2\frac{\alpha}{2} - \cancel{\text{sen}^2\alpha/2} + i\,2\cos\alpha/2\,\sin\alpha/2$$

$$= 2\cos^2\alpha/2 + i\cdot 2\cos\alpha/2\,\sin\alpha/2$$

$$= 2\cos\alpha/2\,(\cos\alpha/2 + i\sin\alpha/2)$$

$$= 2\cos\alpha/2\cdot e^{i\alpha/2}$$

$$\Rightarrow r = 2\cos\frac{\alpha}{2}, \quad \beta = \alpha/2.$$

Con esto, tenemos que $(1+\cos\alpha + i\,\text{sen}\,\alpha)^n = \left(2\cos\frac{\alpha}{2}\, e^{i\alpha/2}\right)^n$

$$= 2^n\cos^n\left(\frac{\alpha}{2}\right) e^{i\frac{\alpha}{2}n}$$

Con lo anterior, tenemos que:

$$A + iB = 2^n\cos^n\left(\frac{\alpha}{2}\right)\cdot e^{i\frac{\alpha}{2}n}$$

$$\Rightarrow Re(A+iB) = \mathcal{R}\left(2^n\cos^n\left(\frac{\alpha}{2}\right) e^{i\frac{\alpha}{2}n}\right)$$

$$\Rightarrow A = 2^n\cos^n\left(\frac{\alpha}{2}\right)\cdot\cos^n\left(\frac{\alpha}{2}n\right) \quad //$$

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P3

a) Resuelva la ecuación $z^2 = -4i$.

Dem

Tenemos $z^2 = -4i = 4\cdot(-i) = 4\cdot e^{\frac{3\pi}{2}\cdot i}$

Usando la formula, Tenemos las soluciones: para $n = 2$

$$z_k = 4^{1/n}\cdot e^{i\left(\frac{\frac{3\pi}{2}+2k\pi}{n}\right)}$$

$$= \sqrt{4}\, e^{i\left(\frac{3\pi}{4}+k\pi\right)}$$

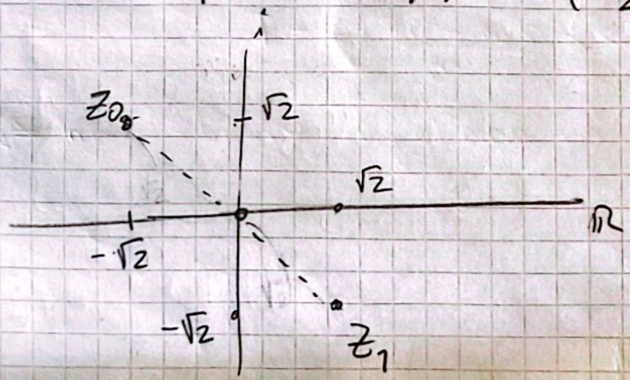
$$= 2\, e^{i\left(\frac{3\pi}{4}+k\pi\right)}$$

$$\Rightarrow z_0 = 2\cdot e^{i\frac{3\pi}{4}}, \quad z_1 = 2\cdot e^{i\left(\frac{3\pi}{4}+\pi\right)} = 2\cdot e^{i\frac{7\pi}{4}}$$

En Cartesiano:

$$z_0 = 2\left(\cos\frac{3\pi}{4} + i\,\text{sen}\,\frac{3\pi}{4}\right) = 2\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \sqrt{2}(-1+i)$$

$$z_1 = 2\cdot\left(\cos\frac{7\pi}{4} + i\,\text{sen}\,\frac{7\pi}{4}\right) = 2\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = \sqrt{2}(1-i)$$



b)

las raíces de la unidad cumplen

$$z^n = 1 = e^0 \rightarrow z_k = e^{i\left(\frac{0+2k\pi}{n}\right)}, \quad k \in \{0, \ldots, n-1\}$$

$$= e^{i\frac{2k\pi}{n}}, \quad k \in \{0, \ldots, n-1\}$$

además, las raíces de 1 cumplen:

$$\sum_{k=0}^{n-1} z_k = 0$$

Con esto: $\displaystyle\sum_{k=0}^{n-1} z_k = 0 \Rightarrow Re\left(\sum_{k=0}^{n-1} z_k\right) = Re(0)$

$$\Rightarrow \sum_{k=0}^{n-1} Re(z_k) = 0$$

Tenemos: $Re(z_k) = \cos \frac{2k\pi}{n}$

$$\Rightarrow \sum_{k=0}^{n-1} \cos \frac{2k\pi}{n} = 0$$

$$\Rightarrow \cos 0 + \cos \frac{2\pi}{n} + \ldots + \cos \frac{2(n-1)\pi}{n} = 0$$

$$\Rightarrow \cos \frac{2\pi}{n} + \ldots + \cos \frac{2(n-1)\pi}{n} = -1$$

Para la suma de sen es análogo usando $Im(z_k)$