Pauta.

P1

a) $\mathcal{L}(\text{sen}t * \text{sen}t) = \mathcal{L}(\text{sen}t) \cdot \mathcal{L}(\text{sen}t) = \frac{1}{(s^2+1)} \cdot \frac{1}{(s^2+1)}$

$$= \frac{1}{(s^2+1)^2}$$

$\boxed{0.5\,pt}$

Calculemos la convolución

$$\text{sen}t * \text{sen}t = \int_0^t \text{sen}u \ \text{sen}(t-u)\,du$$

$\text{sen}(t-u) = \text{sen}t \cos u - \cos t \ \text{sen}u$

$$\int_0^t \text{sen}u \ (\text{sen}t \cos u - \cos t \ \text{sen}u)\,du$$

$$= \text{sen}t \int_0^t \text{sen}u \cos u \, du - \cos t \int_0^t \text{sen}^2 u \, du \quad \boxed{0.2\,pt}$$

$$= \text{sen}t \int_0^t \frac{\text{sen}\,2u}{2}\,du - \cos t \int_0^t \frac{1-\cos 2u}{2}\,du$$

$$= \text{sen}t\left(-\frac{\cos 2t}{4} + \frac{1}{4}\right) - \frac{1}{2}t\cos t + \cos t \left.\frac{\text{sen}\,2t}{4}\right|_0^t$$

$\boxed{0.3\,pt}$

$$= \text{sen}t\left(-\frac{\cos 2t}{4}\right) + \frac{1}{4}\text{sen}t - \frac{1}{2}t\cos t + \cos t\,\frac{\text{sen}\,2t}{4}$$

$$= \frac{1}{4}\text{sen}(2t - t) + \frac{1}{4}\text{sen}t - \frac{1}{2}t\cos t = \frac{1}{2}\text{sen}t - \frac{1}{2}t\cos t \quad \boxed{0.5\,pt}$$

(b) $$\frac{d}{ds}\left(\frac{1}{(s^2+1)}\right) = \frac{-2s}{(s^2+1)^2}$$

$$\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{d}{ds}\mathcal{L}(\text{sent})$$ $\boxed{0.5\,pt}$

$$\frac{d}{ds}\mathcal{L}(\text{sent}) = \mathcal{L}(-t\,\text{sent})$$ (0.5) pt

Por lo tanto
$$\mathcal{L}(-t\,\text{sent}) = \frac{d}{ds}\frac{1}{s^2+1} = \frac{-2s}{(s^2+1)^2}$$
$\boxed{0.5\,pt}$

$$\Rightarrow \mathcal{L}\left(\frac{t\,\text{sent}}{2}\right) = \frac{s}{(s^2+1)^2}$$

c) (i) $$ty'' - 2y' + ty = 0 \quad /\mathcal{L}$$

$$\mathcal{L}(ty'') - 2\mathcal{L}(y') + \mathcal{L}(ty) = 0$$

$$\mathcal{L}(ty'') = -\frac{d}{ds}\mathcal{L}(y'')$$ $\boxed{0.3\,pt}$

$$\mathcal{L}(ty) = -\frac{d}{ds}\mathcal{L}(y)$$

$$-\frac{d}{ds}\left[s^2\mathcal{L}(y) - s\cdot 1 - 0\right] - 2\left[s\mathcal{L}(y) - 1\right] - \frac{d}{ds}\mathcal{L}(y) = 0$$
$\boxed{0.3}$

$$-2s\mathcal{L}(y) - s^2 \frac{d}{ds}\mathcal{L}(y) + 1 - 2s\mathcal{L}(y) + 2 - \frac{d}{ds}\mathcal{L}(y) = 0$$

$\boxed{0.1\ pt}$

$$Y(s) = \mathcal{L}(y)$$

$$-(s^2+1)\,Y'(s) - 4s\,Y(s) + 3 = 0$$

$$Y'(s) + \frac{4s}{(s^2+1)}\,Y(s) = \frac{3}{(s^2+1)}$$

$\boxed{0.3\ pt}$

(ii) Resolvamos la ecuación usando factor integrante ya que es una ecuación lineal no homogénea.

Para eso calculamos

$$e^{\int \frac{4s}{(s^2+1)} ds} = e^{2\int \frac{2s}{s^2+1} ds} = e^{2\ln(s^2+1)} = e^{\ln(s^2+1)^2} = (s^2+1)^2$$

$\boxed{0.4\ pt}$

Así obtenemos

$$\frac{d}{ds}\left(Y(s)(s^2+1)^2\right) = 3(s^2+1)$$

$$Y(s) = \left(s^3 + 3s + C\right) / (s^2+1)^2$$

$\boxed{0.6\ pt}$

(iii) Usando la indicación

$$Y(s) = \frac{s(s^2+1)+2s}{(s^2+1)^2} + \frac{C}{(s^2+1)^2}$$

$$= \frac{s}{s^2+1} + \frac{2s}{(s^2+1)^2} + \frac{C}{(s^2+1)^2}$$ $\boxed{0.2\ pt}$

Usando la parte (a) y (b)

$\boxed{0.8\ pt}$

$$y(t) = \cos t + t\,\mathrm{sen}\,t + C\left(\frac{1}{2}\mathrm{sen}\,t - \frac{1}{2}t\cos t\right.$$

P2

(a) Usamos Laplace

$$\mathcal{L}(y'') + \mathcal{L}(y) = e^{-\pi s} - e^{-2\pi s}$$

$$s^2\mathcal{L}(y) - s\cdot 0 - 1 + \mathcal{L}(y) = e^{-\pi s} - e^{-2\pi s} \quad \boxed{1\ pt}$$

$$(s^2+1)\,\mathcal{L}(y) = 1 + e^{-\pi s} - e^{-2\pi s}$$

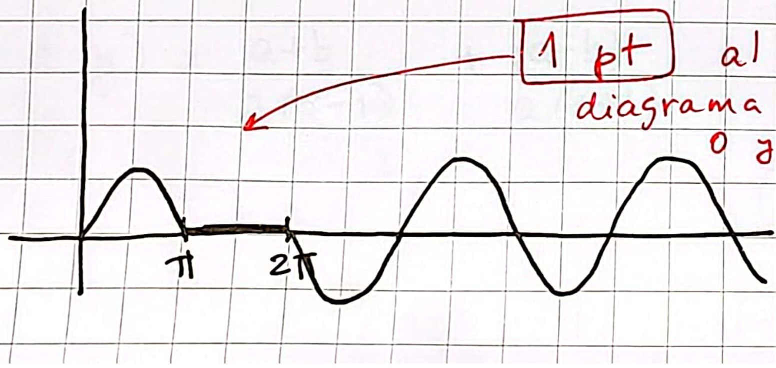
$$\mathcal{L}(y) = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1} - \frac{e^{-2\pi s}}{s^2+1} \quad \boxed{0.5\,pt}$$

$$y(t) = \text{sen}\,t + H(t-\pi)\,\text{sen}(t-\pi) - H(t-2\pi)\,\text{sen}(t-2\pi)$$

$$\text{Sen}(t-\pi) = \text{sen}\,t\,\underbrace{\cos\pi}_{-1} - \cancel{\text{sen}\,\pi}_{0}\cos t = -\,\text{sen}\,t$$

$$y(t) = \text{sen}\,t + H(t-\pi)(-\,\text{sen}\,t) - H(t-2\pi)\,\text{sen}\,t$$

$$= \text{sen}\,t - H(t-\pi)\,\text{sen}\,t - H(t-2\pi)\,\text{sen}\,t \quad \boxed{0.5\ pt}$$



$\boxed{1\ pt}$ al menos diagrama en tre 0 y 3π

$H(t-\pi) = 1$
$t > \pi$
$\text{sent} - \text{sent} = 0$

$t > 2\pi \qquad H(t-\pi) = H(t-2\pi) = 1$

$\text{sent} - \text{sent} - \text{sent}$

(b) $$y'' - y = g(t)$$

$$y(0) = a \qquad y'(0) = b.$$

$$\mathcal{L}(y'') - \mathcal{L}(y) = \mathcal{L}(g)$$

$$s^2 \mathcal{L}(y) - s\cdot a - b - \mathcal{L}(y) = \mathcal{L}(g)$$

$$(s^2-1)\,\mathcal{L}(y) = sa + b + \mathcal{L}(g)$$

$$\mathcal{L}(y) = \frac{sa+b}{s^2-1} + \frac{\mathcal{L}(g)}{s^2-1} \qquad \boxed{1\ pt}$$

$$\frac{1}{s^2-1} = \frac{A}{s-1} + \frac{B}{s+1} \qquad (A+B) = 0 \qquad A - B = 1 \qquad -2B = 1\ ;\ B = \frac{-1}{2} \quad A = \frac{1}{2}$$

$$\frac{sa+b}{s^2-1} = \frac{A}{s-1} + \frac{B}{s+1}$$

$$= A(s+1) + B(s-1)$$

$$A + B = a$$
$$A - B = b$$

$$2A = a+b$$
$$A = \frac{a+b}{2} \qquad B = \frac{(a-b)}{2}$$

$$\mathcal{L}(y) = \frac{a+b}{2(s-1)} + \frac{(a-b)}{2(s+1)} + \mathcal{L}(g)\left(\frac{1}{s-1} - \frac{1}{s+1}\right)\frac{1}{2} \qquad \boxed{1\ pt}$$

$$y = e^{t}\left(\frac{a+b}{2}\right) + e^{-t}\left(\frac{a-b}{2}\right) + \left(g * \left(\frac{e^{t}-e^{-t}}{2}\right)\right) \leftarrow \text{así} \quad \text{ó} \quad \boxed{1\ pt}$$

$$= e^{t}\left(\frac{a+b}{2}\right) + e^{-t}\left(\frac{a-b}{2}\right) + \int_0^t g(u)\, \operatorname{senh}(t-u)\, du \quad \checkmark \text{así}$$